Settled sequential equilibrium (preliminary)

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Wanted!

A solution concept for finite extensive-form games that is

1. compatible with individual dynamic programming (one decision-maker for each player)
2. consistent with Bayesian rationality [Savage, 1954]
3. invariant under the Thompson (1952) transformations [Elmes and Reny, 1994]
4. robust against "strategic uncertainty" (beliefs that attach less than unit probability to others' adherence)
5. a potential "convention" in a population setting à la Nash's (1950) mass-action interpretation [Myerson & Weibull, 2015]
6. non-empty valued

No established solution concept meets all demands:

- Nash equilibrium (Nash, 1950) fails 1-5
- Subgame-perfect equilibrium (Selten, 1965) fails 1-5
- Extensive-form perfect equilibrium (Selten, 1975) fails 1, 3 and 5
- Sequential equilibrium (Kreps and Wilson, 1982) fails 2-5
- Quasi-perfect equilibrium (van Damme, 1984) and strategic stability (Kohlberg & Mertens, 1986, Mertens, 1989) fail 5

$$\varnothing \neq EFPE \subset SE \subset SPE \subset NE$$

$$\varnothing \neq QPE \subset SE$$

1 Sequential equilibrium

This is, arguably, the most well-known and used solution concept for finite extensive-form games with perfect recall.

Definition 1.1 (Kreps and Wilson, 1982) *A* **belief system** *in an extensive-form game* Γ *is a function* $\mu : A \diagdown A_\omega \rightarrow [0, 1]$ *such that*

$$\sum_{a \in D} \mu(a) = 1 \qquad \forall D \in \mathcal{D}$$

Definition 1.2 (Kreps and Wilson, 1982) *A belief system* μ *is* **consistent** *with a behavior-strategy profile* y *in* Γ *if there exist interior behavior-strategy profiles* $y^t \rightarrow y$ *such that* $\mu\left(a \mid y^t\right) \rightarrow \mu(a)\ \forall a \in A \diagdown A_\omega$, *where* $\mu\left(\cdot \mid y^t\right) : A \diagdown A_\omega \rightarrow [0, 1]$ *is the belief system induced by Bayes' law from* y^t.

Definition 1.3 *A behavior-strategy profile y^* is* **sequentially rational** *under a belief system μ if for every player i and information set D:*

$$y_i^* \in \arg\max_{y_i \in Y_i} \sum_{a \in D} \mu(a)\, \pi_{ia}\left(y_i, y_{-i}^*\right)$$

Definition 1.4 (Kreps and Wilson, 1982) *A behavior-strategy profile y^* is a* **sequential equilibrium** *if there exists a consistent belief system μ under which y^* is sequentially rational.*

2 The Thompson transformations

Elmes and Reny (1994) identify 3 transformations:

1. ADD: add a node to a player's information set so that the player's choice at the information set will not affect any player's payoff in case play would reach the added node

2. COA: coalesce two consecutive singleton information sets for a player to one decision node

3. INT: interchange the order of moves between two players who are not informed of each others' moves

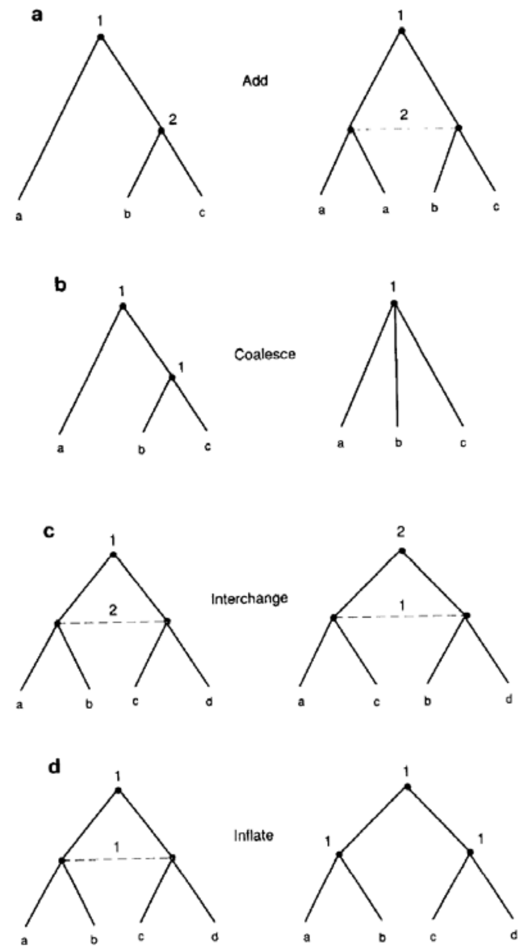
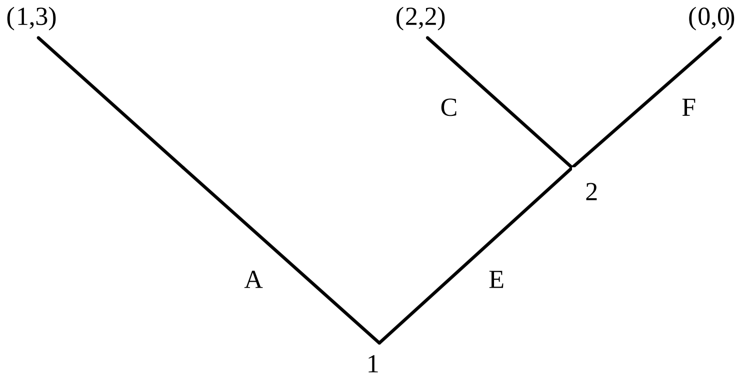
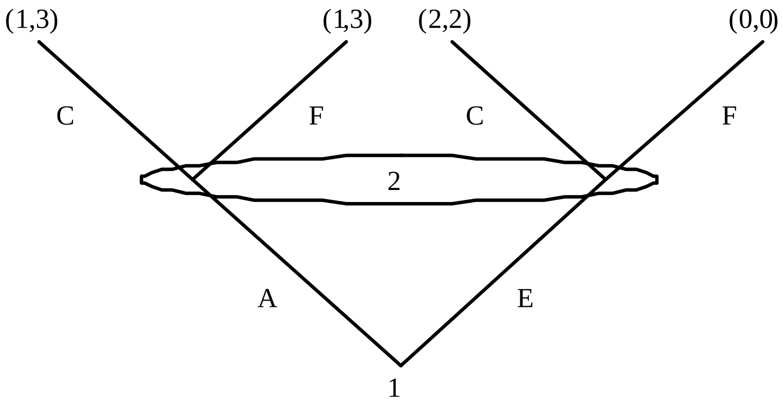


FIGURE 3.1

Example:



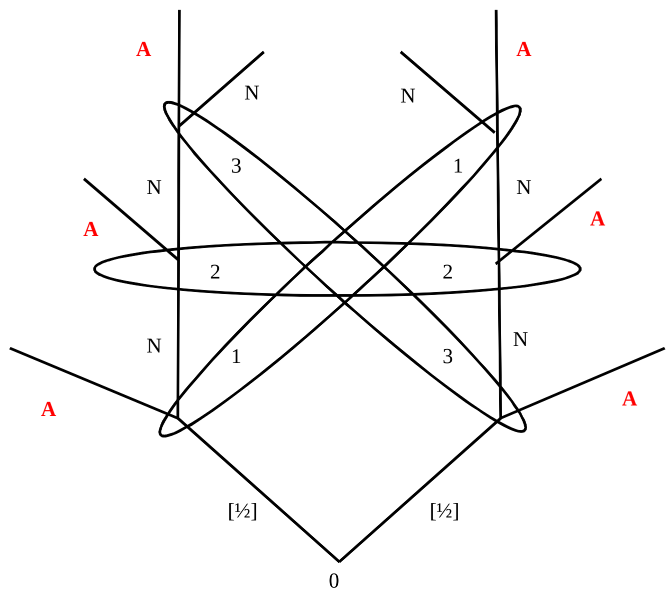
Game 1



Game 2

- SE is not invariant under the Thompson transformations, as seen in Games 1 and 2

- SE is not compatible with Bayesian rationality in all games. In the following example, sequential rationality and consistency together require a form of "cognitive dissonance" on behalf of the player (Kreps and Ramey, 1988):



Game 3

- Moreover, sequential equilibria are in general not robust to strategic uncertainty

Definition 2.1 *A behavior-strategy profile* y^* *is a* robust sequential equilibrium *if every neighborhood of* y^* *contains some interior behavior strategy* y' *such that, for every player* i *and information set* $D \in \mathcal{D}_i$,

$$y_i^* \in \arg\max_{y_i \in Y_i} \sum_{a \in D} \mu\left(a \mid y'\right) \hat{u}_{ia}\left(y_i, y'_{-i}\right) \qquad (1)$$

where $\mu\left(\cdot \mid y'\right) : A \diagdown A_\omega \rightarrow [0,1]$ *is the belief system induced by Bayes' law from* y'.

- This is equivalent with van Damme's (1984) definition of quasi-perfect equilibrium

3 Equilibria as potential social conventions

Myerson and Weibull (2015): "Tenable strategy blocks and settled equilibria"

Game 4:

	L	R
L	$1, 1$	$0, 0$
R	$0, 0$	$1, 1$

- The mixed NE is arguably not tenable as a social convention in a population setting, but is perfect, strategically stable and robust sequential equilibrium (in the associated simultaneous-move game)

[Exceptional solution concepts: *evolutionary stability* (Maynard Smith & Price, 1973) and *persistent equilibrium* (Kalai and Samet, 1984)]

The same is true for the following basic signalling game (Balkenborg, Hofbauer & Kuzmics, 2014, "The refined best-response correspondence in normal form games"):

Example 3.1 *A sender-receiver game with two equally likely states of nature,* A *and* B*:*

(1) The sender observes the state of nature and sends one of two messages, a *or* b*, to the receiver*

(2) Having received the message, the receiver takes one of two actions, α *or* β

(3) Both players receive payoff 2 *if action* α *(*β*) is taken in state* A *(*B*).*

Otherwise they receive zero

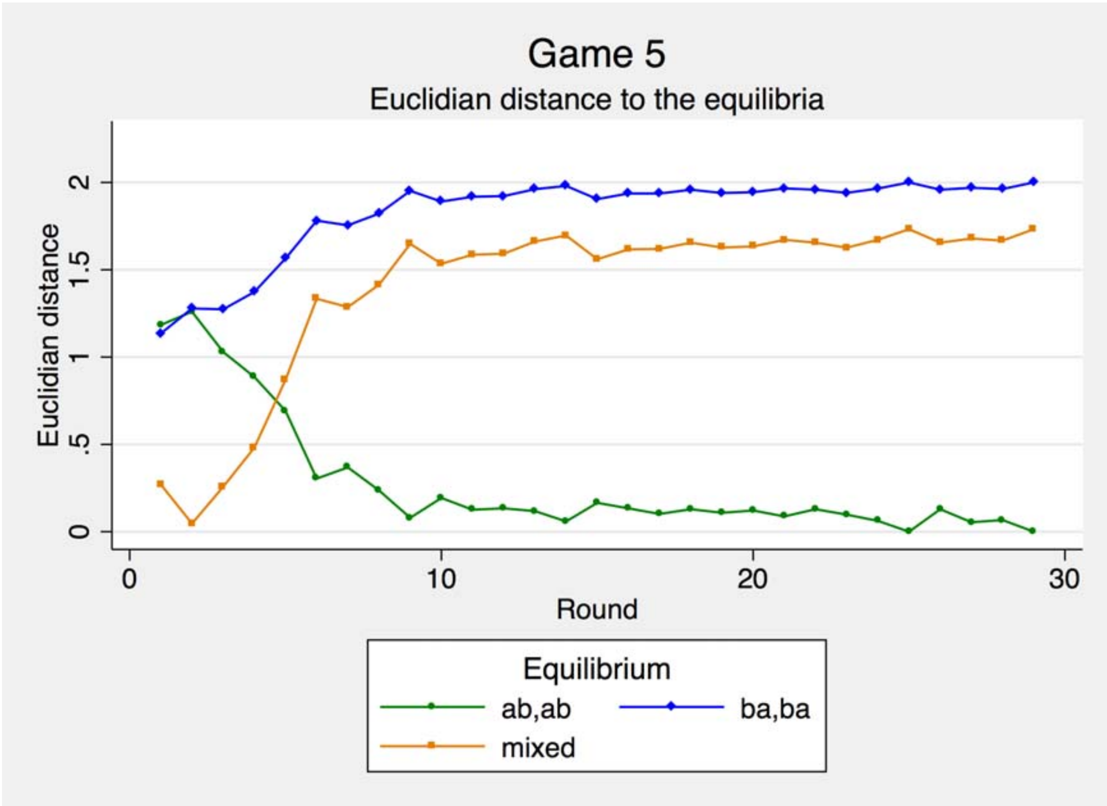
Game 5:

	$\alpha\alpha$	$\alpha\beta$	$\beta\alpha$	$\beta\beta$
aa	$1,1$	$1,1$	$1,1$	$1,1$
ab	$1,1$	$2,2$	$0,0$	$1,1$
ba	$1,1$	$0,0$	$2,2$	$1,1$
bb	$1,1$	$1,1$	$1,1$	$1,1$

Infinitely many NE (6 pure and a continuum of mixed) that are perfect, even proper and KM strategically stable (as singletons)

The two NE $s^* = (ab, \alpha\beta)$ *and* $s^{**} = (ba, \beta\alpha)$ *stand out in a population setting with recurrently play, since they are as if there exists a language, a convention*

Experiment (with Ph.D. student Eskil Forsell, 2015): pairwise anonymous random matching in a population of size 40, 29 sessions during 5 days. Public population statistics after each session. Game presented in normal form, with randomized neutral names given to pure strategies. Payoffs are numbers of lottery tickets, in a lottery after all sessions have been run, with a prize of 10,000 SEK (1,100 euros).



Myerson and Weibull (2015): "Tenable strategy blocks and settled equilibria"

- Formalize potential conventions as strategy blocks, nonempty subsets of pure strategies for each player role in the game

- and look for strategy blocks that are both externally and internally stable:

  - external stability: no player should be able to gain by deviating from the block when others are likely to act conventionally and rationally within the block (but may occasionally act unconventionally and/or irrationally)

  - internal stability: the block should not contain any subblock that is externally stable

Conventionality, rationality & irrationality

Consider any given strategy block as a candidate for a convention. Individuals are of different *types*.

- The *conventional* type considers precisely his/her block strategies

- Unconventional types may be more or less rational (attentive, aware):

  - The fully *rational* type considers *all* available strategies

  - An unconventional type is *less rational* than another if it only considers a proper *subset* of the strategies considered by the other type

- Each individual's type is his or her private information

- We define two varieties of "stable blocks": *coarsely* and *finely tenable*, and two associated equilibrium notions, *coarsely* and *finely settled equilibria*, equilibria with support in minimal such (coarsely resp. finely) tenable blocks

  – The two properties, coarse and fine, *agree* on generic normal-form games

  – ... but *differ* from all established solution concepts on an open set of games

Facts:

1. The maximal block S is coarsely and finely tenable

2. Any strict equilibrium, viewed as a singleton block, is coarsely tenable

Proposition 3.1 *If T is coarsely tenable, then the NE of the block game G_T are precisely the NE of G that have support in T.*

- "Oblivion of unconventional strategies is costless"

Definition 3.1 *A* **coarsely settled equilibrium** *is any Nash equilibrium that has support in a minimal coarsely tenable block.*

Facts:

1. The mixed NE of Game 4 is *not* coarsely settled
2. In Game 5 only $s^* = (ab, \alpha\beta)$ and $s^{**} = (ba, \beta\alpha)$ are coarsely settled

4 Fully settled equilibria

- Coarse tenability imposes no constraint on which player types, other than the conventional ones, are more likely; robustness is required under *any* type distribution that puts sufficient probability on the conventional types

- Fine tenability requires that the type distribution places a lot probability on conventional types, and, among unconventional types attaches much more probability to a “more rational” (or ”more aware”) type than to a “less rational” (or ”less aware”)

Proposition 4.1 *Each finely tenable block contains the support of a proper equilibrium.*

Definition 4.1 *A* **finely settled equilibrium** *is any proper equilibrium that has support in some minimal finely tenable block.*

- A coarsely settled equilibrium need not be finely settled, and a finely settled equilibrium need not be coarsely settled

Definition 4.2 *A* **fully settled equilibrium** *is any equilibrium that is both coarsely and finely settled.*

Proposition 4.2 *Every finite game has at least one fully settled equilibrium.*

5 Settled sequential equilibrium

Definition 5.1 *Pure strategies* $s_i', s_i'' \in S_i$ *are* **payoff equivalent** *if* $\pi_j\left(s_i', s_{-i}\right) = \pi_j\left(s_i'', s_{-i}\right)$ *for all pure-strategy profiles* s *and players* j*.*

- For each player $i \in I$ and pure strategy $s_i \in S_i$ let $[s_i] \subseteq S_i$ denote the set of pure strategies s_i' that are payoff equivalent with s_i.

Definition 5.2 *The* (purely) **reduced normal form** *representation* G^o *of a normal-form game* G *is the normal-form game in which for each player* i *the pure strategies* s_i^o *are the equivalence classes* $[s_i]$ *in* G *and where* π^o *is the accordingly adapted payoff function.*

Definition 5.3 *A behavior-strategy profile in a finite extensive-form game* Γ *is* **payoff-equivalent** *with a mixed-strategy profile in the reduced normal form of* Γ *if they induce the same probability distribution over payoff profiles.*

Definition 5.4 *A* **settled sequential equilibrium** *in a finite extensive-form game* Γ *is any robust sequential equilibrium that is payoff-equivalent with some fully settled equilibrium of the reduced normal form of* Γ*.*

Found!

A solution concept that is

1. compatible with individual dynamic programming

2. consistent with Bayesian rationality [Savage, 1954]

3. invariant under the Thompson (1952) transformations [Elmes and Reny, 1994]

4. robust against strategic uncertainty (beliefs that attach less than unit probability to others' adherence)

5. a potential convention in a population setting à la Nash's (1950) mass-action interpretation [Myerson & Weibull, 2015]

6. non-empty valued

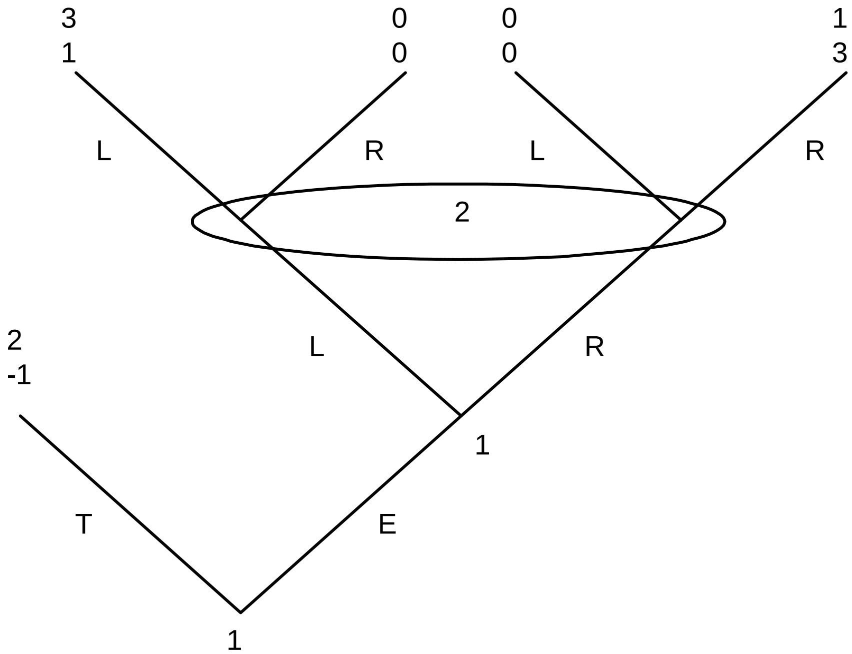
Rests upon:

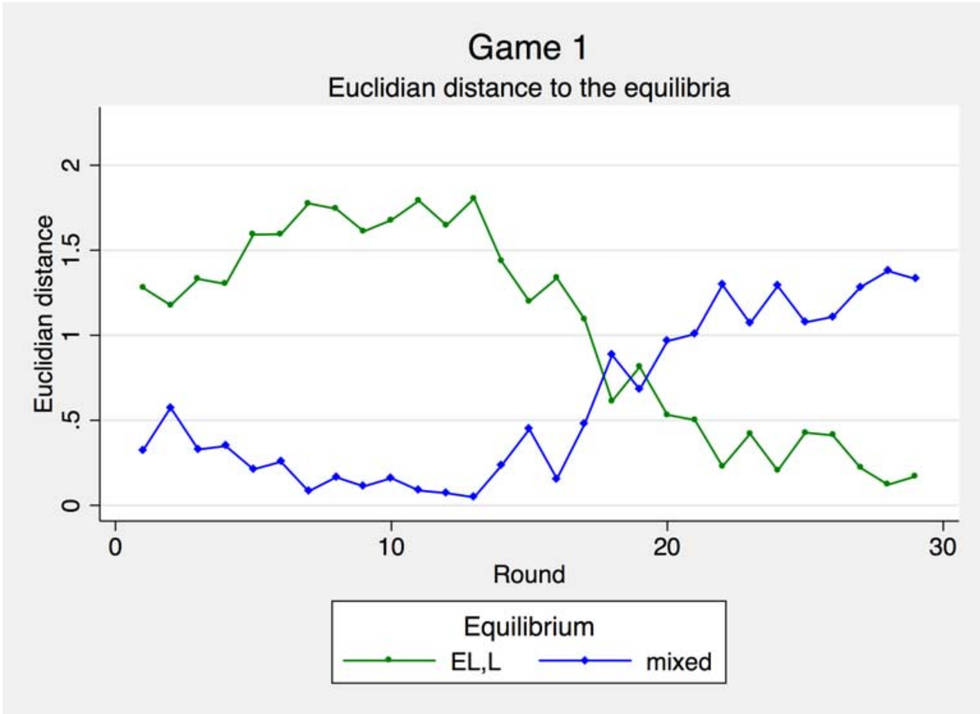
Theorem 5.1 (van Damme, 1984) *Let G be a finite normal-form game and let x^* be a proper equilibrium of G. In every finite extensive-form game Γ with perfect recall and with G as its normal form, there exists a quasi-perfect equilibrium y^* of Γ that is payoff-equivalent with x^*.*

Theorem 5.2 (Elmes and Reny, 1994) *If Γ and Γ' are finite extensive-form games with perfect recall that have the same reduced normal form, then there exists a finite set of extensive-form games, $\Gamma_1, ..., \Gamma_k$, each with perfect recall, such that (a) $\Gamma_1 = \Gamma$ and $\Gamma_k = \Gamma'$, and (b) two consecutive games in this set differ only by one of the transformations ADD, COA or INT.*

THE END

Other experimental results: Outside option and forward induction:




Game 1
Euclidian distance to the equilibria
Euclidian distance
0
.5
1
1.5
2
0
10
20
30
Round
Equilibrium
EL,L
mixed