A characterization of the sets of equilibrium payoffs of finite games

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Sets of equilibrium payoffs of finite games

What is known about the set $E \subset \mathbb{R}^N$ of (mixed) equilibrium payoffs of some (one-shot) finite game with N players ?

- It is nonempty
- It is compact
- It is semialgebraic : finite union and intersection of sets each of the form $\{e \in \mathbb{R}^N, P(e_1, \cdots, e_N) < 0\}$ or $\{e \in \mathbb{R}^N, P(e_1, \cdots, e_N) \leq 0\}$.
- "Generically" it is finite and with odd cardinality.

Main problem

Answer to the question : $E \subset \mathbb{R}^N$ is the set of the equilibrium payoffs of some finite game with N players iff ...

- $N = 1$: iff E is a singleton.
- $N = 2$: iff E is the (nonempty) finite union of sets of the form $[a,b] \times [c,d]$ (Lehrer Solan Viossat '11)

What happens when $N \geq 3$?

Related literature

"Characterization" of sets of equilibria of finite games:

- Datta '03 : any real algebraic variety is isomorphic to the set of completely mixed equilibrium of some 3 player game, and to the set of completely mixed equilibrium of some M player binary game.
- Balkenborg-Vermeulen '14 : any nonempty connected compact semi-algebraic set is homeomorphic to one connected component of the set of equilibria of binary common interest game with payoffs 0 or 1.
- Levy '15, Viossat-V '15 : any nonempty compact semi-algebraic set is the projection of the set of equilibria of some game with M additional binary players.
- Levy '15 : any nonempty compact semi-algebraic set is the projection of the set of equilibria of some game with 3 additional nonbinary players.

Answer to our problem

Proposition

If $N \geq 3$, $E \subset \mathbb{R}^N$ is the set of equilibrium payoffs of some finite N-player game iff E is nonempty compact and semialgebraic.

Only the "if" part has to be proven.

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Preliminaries

- Fact 1 : any closed semialgebraic set in $\mathbb{R}^n$ is the finite union and intersection of sets of the form $\{e \in \mathbb{R}^N, P(e_1, \cdots, e_N) \leq 0\}$.
- Fact 2 : It is enough to prove the result when $E \subset]0, \varepsilon(N)[$ (for some $\varepsilon(N) > 0$ to be fixed).
  Proof :
  - applying an affine transformation to all the payoffs of some game does not change the set of equilibria.
  - applying an affine transformation to a nonempty closed semialgebraic set yields a nonempty closed semialgebraic set.

Preliminaries

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Preliminaries

Fact 3 : If E_1 and E_2 are sets of Nash equilibrium payoffs, then so is $E_1 \cup E_2$.
Proof : based on a construction in Lehrer Solan Viossat '11
For $N = 3$ for example, let A^i and B^i be the matrix of payoffs of player i in each game, adding irrelevant actions if necessary so that each game has the same number of actions, and consider the game

$$\begin{pmatrix} A^1,A^2,A^3 & -C,A^2,-C \\ A^1,-C,-C & B^1,B^2,-C \end{pmatrix} \qquad \begin{pmatrix} -C,-C,A^3 & -C,B^2,B^3 \\ B^1,-C,B^3 & B^1,B^2,B^3 \end{pmatrix}$$

for some large constant C.

Main Lemma

So it is enough to show the result when
$E = \cap_{m=1}^{M} \{e \in \mathbb{R}^N, P_k(e_1, \cdots, e_N) \leq 0\}$ is a nonempty set in $]0, \varepsilon(N)[$.

We claim that it is now enough to prove

Lemma

Let E be such a set. Then there exists an N player game, with a particular action X_*^i for each player, in which
- There exists an equilibrium which gives a probability e_i to X_*^i for every i iff $e = (e_1, \cdots, e_n) \in E$
- all equilibria have a payoff of 0

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Proof of the claim

Consider such a game and add 1 to the payoff of each player i iff player $i-1$ plays X_*^{i-1}.

- This does not change the set of equilibria
- Hence the set of equilibrium payoffs of this new game is $\{(e_N, e_1, \cdots e_{N-1}) | e \in E\}$
- One just has to relabel the players.

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Notations

- We assume than $N = 3$. $i+1$ and $i-1$ are to be understood mod 3.
- We will drop the non emptiness assumption on E and prove the lemma for $E' = E \cup \{z\}$ where $z \in]0, \varepsilon[^3$. Apply it to $z \in E$ for the lemma.
- Actions will be denoted by uppercase (e.g. X_*^i) and probabilities by the corresponding lowercase (e.g. x_*^i).
- Payoff of an action of Player i will be given as "multiaffine" maps of the x^j for $j \neq i$ for example $g^1(X_*^1) = x_*^2 + 2x_*^2 x_*^3$.

Set of actions

Each player i has two family of actions:

- Actions denoted with letter X, including some special action X^i_*. Called "unknowns". Give a payoff 0 (except X^i_*), are all played with positive probability in all equilibria (but one).
- Actions denoted with letter Y, including some special action Y^i_*. Called "constraints". Not played at equilibrium (except Y^i_* in one equilibrium), which gives information on the probabilities x^i. Denote by $s^i = \sum y^i$ the probability that player i play some constraint.

We will prove that the set of equilibria is exactly :

- For each $e \in E$, an equilibria with $s^i = 0$ and $x^i_* = e^i$ for all i. Called "nice equilibria".
- An additional "sporadic" equilibria with $s^i > 0$ for all i and $x^i_* = z^i$.

The unknowns

Let D be a bound for the total degree of each P_m.

- X^i_* with payoff to be determined later on.
- $X^i_{j,k,l}$ for $D \geq j+k+l \geq 1$, with $g^i(X^i_{j,k,l}) = 0$.
  Role : we'll have $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ in nice equilibria.
- X^i_0 with $g^i(X^i_0) = 0$.
  Role : dump of probabilities so that profiles lie in the simplex.

The initialization constraints

Role : to ensure that at nice equilibria $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l = 1$.
8 strategies for player 1 : 4 of payoffs $\pm(x^p_{0,1,0} - x^2_*)$ for $p = 2$ or 3, and 4 of payoffs $\pm(x^p_{0,0,1} - x^3_*)$ for $p = 2$ or 3.
Similar strategies for player 2 and 3.

The induction constraints

Role : to ensure that at nice equilibria $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l \geq 2$.
For any $j+k+l \geq 2$, if $j \geq 1$ add 2 strategies of player i with payoff $\pm(x^{i-1}_{j,k,l} - x^{i-1}_{j-1,k,l} x^{i+1}_{1,0,0})$.
Similar strategies if $j = 0$ but $k \geq 1$, or if $j = k = 0$ and $l \geq 1$.

The semialgebraic constraints

Role : to ensure that at nice equilibria $(x_*^1, x_*^2, x_*^3) \in E$.
For any m, associate to

$$P_m(e_1, e_2, e_3) = \sum_{0 \le j+k+l \le D} c_m^{j,k,l} e_1^j e_2^k e_3^l.$$

an action of player i with payoff

$$c_m^{0,0,0} + \sum_{1 \le j+k+l \le D} c_m^{j,k,l} x_{j,k,l}^{i-1}.$$

The two particular strategies

- K is some large constant
- $g^i(Y^i_*) = K(1 - 2x_0^{i-1})$
- $g^i(X^i_*) = K\frac{s^{i-1}}{1-z^{i-1}}$
- Role : to ensure that there is only one equilibrium that is not nice.

Nice equilibria : necessary part

Consider an equilibria in which $s^i = 0$ for all i.

- All strategies in the support give a payoff 0, hence the payoff is 0.
- So the payoff of all constraints is nonpositive
- Initialization constraints imply $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l = 1$.
- Induction constraints imply $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l \geq 2$.
- Semialgebraic constraints imply $x_* \in E$. Hence we have proven that there are no nice equilibria for $x_* \notin E$ and at most 1 for $x_* \in E$.

Nice equilibria : sufficient part

If $x_* \in E$,

- Fix $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l \geq 1$
- Thus $x^i_0 = 1 - x^i_* - \sum x^i_{j,k,l}$.
- Fix ε such that $\varepsilon + \sum_{i+j+k \geq 1} \varepsilon^{i+j+k} = \frac{1}{2}$.
- Hence $x^i_0 \geq \frac{1}{2}$ and the profile is well defined
- $g^i(Y^i_*) = K(1 - 2x^{i-1}_0) \leq 0$ so Y^i_* not profitable deviation
- All other constraints are nonprofitable by construction.

Hence there is a nice equilibrium with $x_* \in E$.

Equilibria where $s^i > 0$ for all i

- $g^i(X^i_*) > 0$ hence no unknown (except possibly X^i_*) are part of the support.
- Payoff of constraints is ≤ 1 for initialization, $= 0$ for induction, $= c_m^{0,0,0}$ for semialgebraic, and $K(1-2x^i_0) = K$ for Y^i_*.
- So only X^i_* and Y^i_* may be in the support, provided we fix K large enough.
- $g^i(Y^i_*) = K$, $g^i(X^i_*) = K\frac{1-x^{i-1}_*}{1-z^{i-1}}$ and $y^i_* > 0$.
- Hence $g^i(X^i_*) \leq g^i(Y^i_*)$ and $x^{i-1}_* \geq z^{i-1} > 0$.
- Hence $g^i(X^i_*) = g^i(Y^i_*)$ and $x^* = z$.

Equilibria where $s^i > 0$ for some but not all i

- There is i such that $s^i > 0$ and $s^{i+1} = 0$.
- Without loss of generality $s^1 > 0$ and $s^2 = 0$.
- This implies $x_*^2 = 1$ and thus $x_0^2 = 0$.
- Hence $g(X_*^3) = 0 < K = g(Y_*^3)$ and $s^3 = 1$.
- Thus $g(X_*^1) = \frac{K}{1-z^3} > K$ hence $x_*^1 = 1$ contradiction.

Last fixes

One issue left : the payoff of the sporadic equilibrium is $K \neq 0$.

- Fix : add $\alpha_i y_*^{i-1}$ to all payoff of player i for some $\alpha_i < 0$.
- Won't change the equilibria, nor the payoff of the nice equilibria, and will change the payoff of the sporadic equilibrium.

Other issue : construction works only if N is odd (we use a parity argument in the case of some $s^i > 0$).

- Fix : for $N \geq 6$ define the payoff of x_*^i and y_*^i using two cycles of odd length instead of one cycle of length N.
- Specific fix for $N = 4$.

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Reminder : Hilbert tenth problem

- Problem : "Find an algorithm to determine whether a given multivariate polynomial $P \in \mathbb{Z}[X_1, \cdots X_N]$ has a zero in $\mathbb{Z}^N$.
- MRDP Theorem, Matiyasevich '70 (using previous works by Robinson Davis and Putnam) : this is impossible, there is no such algorithm. "Undecidability of Hilbert tenth problem".
- Still undecidable even if one fix both N and the degree d of P (provided they are larger than explicit bounds).
- Problem is open if one replace "has a zero in $\mathbb{Z}^N$" by "has a zero in $\mathbb{Q}^N$" (Hilbert tenth problem on $\mathbb{Q}$).

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Consequences

Because of our construction, it is impossible to find an algorithm that, given a game with integer pure payoffs, answer to any of these questions:

- Is there an equilibrium in which the payoff of each player is the inverse of an integer ?
- Is there an equilibrium in which the probability that each player plays its first strategy is the inverse of an integer ?

Same impossibility even if one fix both the number of players and of actions (provided they are greater than some explicit bounds).

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Consequences

If Hilbert tenth problem on $\mathbb{Q}$ is also undecidable, then it is impossible to find an algorithm that, given a game with integer payoffs answer to any of these questions:

- Is there an equilibrium in which the payoff of each player is a rational ?
- Is there an equilibrium in which the probability that each player plays its first strategy is a rational ?
- Is there an equilibrium in which all players play all strategies with some rational probability ?

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More questions

- Can we do such a construction with "robust" equilibria (without using weakly dominated strategies for example)?
- If A and B are sets of Nash equilibrium payoffs and $A \cap B$ is nonempty, then it is a set of Nash equilibrium payoff. Direct proof of this ?
- Can we use this to construct stochastic games with weird behavior ?
- What if we add some structure (games with perfect observation for example) ?
- What about the set of payoffs of correlated equilibria ?

Thank you for your attention

Thank you !