Limit value of dynamic zero-sum games with vanishing stage duration

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Abstract

We consider two person zero-sum games where the players control at discrete times t_n of a partition Π of $\mathbb{R}^+$, a continuous time Markov process.

We prove that the limit of the values v_Π exist as the mesh of Π goes to 0.

The analysis covers the cases of :

1) stochastic games (where both players know the state)

2) symmetric no information case.

The proof is by reduction to deterministic differential games.

Introduction

Repeated interactions in a stationary environment have been traditionally represented by repeated games.

An alternative approach is to consider a continuous time process on which the players act at discrete times.

In the first case the number of interactions increases as the weight of each stage goes to zero.

In the second case it increases as the duration of each stage vanishes.

In a repeated game framework one can normalize the evolution of the play using the evaluation and consider a game played on $[0, 1]$ where time t corresponds to the fraction t of the total duration. Each evaluation $\theta = \{\theta_n\}$ (in the original repeated game) thus induces trough the stages of the interaction a partition Π_θ of $[0, 1]$ and vanishing stage weight corresponds to vanishing mesh.

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Tools adapted from continuous time models can be used to obtain convergence results, given a family of evaluations, for the corresponding family of values v_θ, see e.g. Vieille (1992), Sorin (2002), Laraki (2002), Cardaliaguet, Laraki and Sorin (2012).
In the alternative approach there is a given evaluation k on $\mathbb{R}^+$ and one consider a sequence of partitions with vanishing mesh (vanishing stage duration).
In both cases for each given partition the value function exists at the times defined by the partition and the stationarity of the model allows to write a recursive equation. Then one extends the value function to $[0,1]$ (resp. $\mathbb{R}^+$) by linearity and one considers the family of values as the mesh of the partition goes to 0.

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The two main points consists in defining a PDE (E) and proving:
1) that any accumulation point of the family is a viscosity solution of (E) (with an appropriate definition)
2) that (E) has a unique viscosity solution.

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Differential games

The approach of studying the value trough discretization was initiated in Fleming (1957), (1961), (1964), see also Friedman (1971), (1974), Eliott and Kalton (1972).
Z is the state space,
I and J the action sets,
f the dynamics,
g the payoff
k the evaluation function.
Consider a differential game Γ defined on $[0, +\infty)$ by the dynamics:

$$\dot{z}_t = f(z_t, i_t, j_t) \tag{1}$$

and the total payoff:

$$\int_0^{+\infty} g(z_s, i_s, j_s) k(s) ds.$$

Z, I, J subsets of $\mathbb{R}^n$,
I and J compact,
f and g continuous and uniformly Lipschitz in z,
g bounded,
$k : [0, +\infty) \to [0, +\infty)$ Lipschitz with $\int_0^{+\infty} k(s)ds = 1$.

$\Phi^h(z; i, j)$ is the value at time $t + h$ of the solution of (1) starting at time t from z and with play $i_s = i$, $j_s = j$ on $[t, t+h]$.

To define the strategies we have to specify the information: we assume that the players know the initial state, and at time t the previous behavior $(i_s,\ j_s; 0 \leq s < t)$ hence the trajectory of the state $(z_s;\ 0 \leq s < t)$.

1. Deterministic analysis

Let $\Pi = (\{t_n\}, n = 0, ...)$ be a partition of $[0, +\infty)$ with $t_0 = 0, \delta_n = t_n - t_{n-1}$ and $\delta = \sup \delta_n$.

Consider the associate discrete time game γ_Π where on each interval $[t_n, t_{n+1})$ players use constant moves (i_n, j_n) in $I \times J$.

This defines the dynamics.

At time t_{n+1}, (i_n, j_n) is announced thus the next value of the state, $z_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n}; i_n, j_n)$ is known.

The corresponding maxmin w^-_Π (resp. minmax w^+_Π) satisfies the recursive formula:

$$w^-_\Pi(t_n, z_{t_n}) = \sup_I \inf_J \left[\int_{t_n}^{t_{n+1}} g(z_s, i, j) k(s) ds + w^-_\Pi(t_{n+1}, z_{t_{n+1}}) \right] \quad (2)$$

The fonction $w^-_\Pi(., z))$ is extended by linearity to $[0, +\infty)$.

The next results follow from Evans and Souganidis (1984), see also Bardi and Capuzzo-Dolcetta (1996).

Proposition (A1)

The family $\{w^-_\Pi\}$ *is equicontinuous in both variables.*

Theorem (A2)

Any accumulation point of the family $\{w^-_\Pi\}$*, as the mesh* δ *of* Π *goes to zero, is a viscosity solution of:*

$$0 = \frac{d}{dt} w^-(t,z) + \sup_I \inf_J \left[\, g(z,i,j)k(t) + \langle f(z,i,j), \nabla w^-(t,z)\rangle \right]. \tag{3}$$

Theorem (A3)

Equation (3) has a unique viscosity solution.

Crandall and Lions, see Crandall, Ishii and Lions (1992).

Corollary (A4)

The family $\{w^-_\Pi\}$ *converges to some* w^-*.*

Let w_∞^- be the maxmin (lower value) of the differential game Γ played using non anticipative strategies with delay.
From Evans and Souganidis (1984), Cardaliaguet (2010), one obtains:

Theorem (A5)

1) w_∞^- *is a viscosity solution of (3).*
2)

$$w_\infty^- = w^-.$$

Obviously similar properties hold for w_Π^+ and w_∞^+.

Define Isaacs's condition $(\mathscr{I}_0)$ on $I \times J$ by :

$$\begin{aligned} & \sup_I \inf_J [\, g(z,i,j)k(t) + \langle f(z,i,j), p\rangle] \\ = \; & \inf_J \sup_I [\, g(z,i,j)k(t) + \langle f(z,i,j), p\rangle], \quad \forall t \in \mathbb{R}^+, \forall z \in Z, \forall p \in \mathbb{R}^n. \end{aligned}$$

Proposition

Assume condition $(\mathscr{I}_0)$.
Then the limit value exists:

$$w^- = w^+ (= w^-_\infty = w^+_\infty)$$

2. Mixed extension

Given a partition Π we introduce two discrete time games related to γ and played on $X = \Delta(I)$ and $Y = \Delta(Y)$ (set of probabilities on I and J respectively).

2.1. Deterministic moves

The first game is defined as above were X and Y are now the sets of moves (this corresponds to “relaxed controls").

The dynamics f (hence the flow Φ) and the payoff g are defined by the expectation w.r.t. x and y:

$$f(z,x,y) = \int_{I\times J} f(z,i,j)x(di)y(dj)$$

$$g(z,x,y) = \int_{I\times J} g(z,i,j)x(di)y(dj).$$

We consider the associate discrete time game $\overline{\Gamma}_\Pi$ where on each interval $[t_n, t_{n+1})$ players use constant moves $(x_n,\ y_n)$ in $X \times Y$. This defines the dynamics. At time t_{n+1}, (x_n, y_n) is announced and the current value of the state, $z_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n};\ x_n,\ y_n)$ is known.

The maxmin W^-_Π satisfies:

$$W^-_\Pi(t_n, z_{t_n}) = \sup_X \inf_Y \left[\int_{t_n}^{t_{n+1}} g(z_s, x, y)k(s)ds + W^-_\Pi(t_{n+1}, z_{t_{n+1}}) \right].$$

The analysis of the previous paragraph applies, leading to :

Proposition

The family $\{W^-_\Pi\}$ is equicontinuous in both variables.

Theorem

1) Any accumulation point of the family $\{W^-_\Pi\}$, as the mesh δ of Π goes to zero, is a viscosity solution of:

$$0 = \frac{d}{dt}W^-(t,z) + \sup_X \inf_Y \left[g(z,x,y)k(t) + \langle f(z,x,y), \nabla W^-(t,z)\rangle \right] \quad (4)$$

2) The family $\{W^-_\Pi\}$ converges to some W^-.

Similarly let W_∞^- be the maxmin (lower value) of the differential game $\overline{\Gamma}$ played (on $X \times Y$) using non anticipative strategies with delay. Then:

Proposition

1) W_∞^- is a viscosity solution of (4).
2)

$$W_\infty^- = W^-.$$

As above, similar properties hold for W_Π^+ and W_∞^+.

Due to the bilinear extension, Isaacs's condition on $X \times Y$ is now $(\mathscr{I})$:

$$\begin{aligned} & \sup_X \inf_Y [g(z,x,y)k(t) + \langle f(z,x,y), p\rangle] \\ = \; & \inf_Y \sup_X [g(z,x,y)k(t) + \langle f(z,x,y), p\rangle], \quad \forall t \in \mathbb{R}^+, \forall z \in Z, \forall p \in \mathbb{R}^n. \end{aligned}$$

and always holds.

Proposition

The limit value exists:

$$W^- = W^+,$$

and is also the value of the differential game played on $X \times Y$.

Remark that due to $(\mathscr{I})$, (4) can be written as

$$0 = \frac{d}{dt}W(t,z) + \mathtt{val}_{X\times Y} \int_{I\times J} [g(z,i,j)k(t) + \langle f(z,i,j), \nabla W(t,z)\rangle] x(di) y(dj) \tag{5}$$

2.2 Random moves

We define another game $\hat{\Gamma}_\Pi$ where on $[t_n, t_{n+1})$ the moves $(i_n, j_n) \in I \times J$ are constant, chosen at random according to x_n and y_n, and announced at time t_{n+1}. The new state is thus, if $(i_n, j_n) = (i, j)$, $z^{ij}_{t_{n+1}} = \Phi^{\delta_n}(z_{t_n}; i, j)$ and is known.
The next dynamic programming property holds:

Proposition

The game $\hat{\Gamma}_\Pi$ has a value V_Π which satisfies:

$$V_\Pi(t_n, z_{t_n}) \quad = \quad \mathtt{val}_{X \times Y} \mathsf{E}_{x,y}[\int_{t_n}^{t_{n+1}} g(z_s, i, j) k(s) ds + V_\Pi(t_{n+1}, z^{ij}_{t_{n+1}})]$$

and as above:

Proposition

The family $\{V_\Pi(t, z), \Pi\}$ is equicontinuous in both variables.

Moreover one has:

Proposition

1) Any accumulation point of the family $\{V_\Pi\}$*, as the mesh* δ *of* Π *goes to zero, is a viscosity solution of the same equation (5).*
2) The family $\{V_\Pi\}$ *converges to* W*.*

Proof
1) Standard from the recursive equation.
2) The proof of uniqueness was done above. ■

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Proposition

1) Any accumulation point of the family $\{V_\Pi\}$, as the mesh δ of Π goes to zero, is a viscosity solution of the same equation (5).
2) The family $\{V_\Pi\}$ converges to W.

Proof

1) Standard from the recursive equation.
2) The proof of uniqueness was done above. ■

Stochastic games with vanishing stage duration

Assume that the state Z_t follows a continuous time Markov process on $\mathbb{R}^+ = [0, +\infty)$ with values in a finite set Ω.
We study in this section the model were the process Z_t is controlled by both players and observed by both (there is no assumptions on the signals on the actions).
This corresponds to a stochastic game in continuous time analyzed trough a discretization Π.
References include Zachrisson (1964), Tanaka and Wakuta (1977), Guo and Hernandez-Lerma (2003), Prieto-Rumeau and Hernandez-Lerma (2012), Neyman (2013) ...

The process is specified by a transition rate $\mathbf{q} \in \mathscr{M}$: $\mathbf{q}$ is a real continuous map on $I \times J \times \Omega \times \Omega$ with $\mathbf{q}(i,j)[\omega,\omega'] \geq 0$ if $\omega' \neq \omega$ and $\sum_{\omega' \in \Omega} \mathbf{q}(i,j)[\omega,\omega'] = 0$.
The transition is given by:

$$\begin{aligned} \mathsf{P}^h(i,j)[\omega,\omega'] &= Prob(Z_{t+h} = \omega | Z_t = \omega, i_s = i, j_s = j, t \leq s \leq t+h) \\ &= \mathbf{1}_{\{\omega\}}(\omega') + h\,\mathbf{q}(i,j)[\omega,\omega'] + o(h) \end{aligned}$$

thus

$$\dot{\mathsf{P}}^h = \mathsf{P}^h \mathbf{q} = \mathbf{q}\,\mathsf{P}^h$$

and

$$\mathsf{P}^h = e^{h\,\mathbf{q}}.$$

Given a partition $\Pi = \{t_n\}$, the time interval $L_n = [t_{n-1}, t_n[$ (which corresponds to stage n) has duration $\delta_n = t_n - t_{n-1}$.
The law of Z_t on L_n is determined by $Z_{t_{n-1}}$ and the choices (i_n, j_n) of the players at time t_{n-1}, that last for stage n.
In particular, starting from $\zeta_n = Z_{t_{n-1}}$, the law of the new state $\zeta_{n+1} = Z_{t_n}$ is a function of ζ_n, the choices (i_n, j_n) and the duration δ_n.
The payoff at time t in stage n ($t \in L_n \subset \mathbb{R}^+$) is defined trough a map $\mathbf{g}$ from $\Omega \times I \times J$ to $\mathbb{R}$:

$$g_\Pi(t) = \mathbf{g}(Z_t; i_n, j_n)$$

Given a probability density $k(t)$ on $\mathbb{R}^+$ the evaluation along a play is:

$$\gamma_\Pi = \int_0^{+\infty} g_\Pi(t)k(t)dt$$

and this defines the game G_Π.
One considers the asymptotics of the game G_Π as the mesh $\delta = \sup \delta_n$ of the partition vanishes.
Note that here again the “evaluation” $k(t)$ is given and fixed.

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Note that here again the “evaluation” $k(t)$ is given and fixed.

Proposition

The value $v_\Pi(t,z)$ *satisfies the following recursive equation:*

$$\begin{aligned} v_\Pi(t_{n-1}, Z_{t_{n-1}}) &= \mathtt{val}_{X\times Y}\,\mathsf{E}_{x,y}[\int_{t_{n-1}}^{t_n} \mathbf{g}(Z_s,x,y)k(s)ds + v_\Pi(t_n, Z_{t_n})] \\ &= \mathtt{val}_{X\times Y}\,\mathsf{E}_{x,y}[\int_{t_{n-1}}^{t_n} \mathbf{g}(Z_s,x,y)k(s)ds) \\ &\quad + \mathsf{P}^{\delta_n}(x,y)[Z_{t_{n-1}},.]\circ v_\Pi(t_n,.)] \end{aligned}$$

where

$$\mu[z,.]\circ f(\cdot) = \sum_{z'} \mu[z,z']f(z')$$

Proposition

The family of values $\{v_{\Pi,\mathbf{k}}\}_\Pi$ *has at least an accumulation point as* $\bar{\delta}$ *goes to 0.*

Let $w_\Pi(t,\zeta) = \langle \zeta, v_\Pi(t,.)\rangle = \sum_\omega \zeta(\omega) v_\Pi(t,\omega)$.
Define $\mathbf{X} = X^\Omega$ and $\mathbf{Y} = Y^\Omega$ and

$$f(\zeta,\mathbf{x},\mathbf{y})(z) = \sum_{\omega\in\Omega} \zeta(\omega)\mathbf{q}(\mathbf{x}(\omega),\mathbf{y}(\omega))[\omega,z]$$

Proposition

$w_\Pi(t,\zeta)$ *satisfies:*

$$\begin{aligned} w_\Pi(t_{n-1},\zeta_{t_{n-1}}) \quad &= \quad \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\int_{t_{n-1}}^{t_n} \langle \zeta_s, \mathbf{g}(.,\mathbf{x}(.),\mathbf{y}(.))\rangle k(s)ds \\ & \qquad + w_\Pi(t_n,\zeta_{t_n})] \end{aligned}$$

Let $w(t,\zeta) = \langle \zeta, v(t,.)\rangle = \sum_\omega \zeta(\omega) v(t,\omega)$ where $v(t,.)$ is an accumulation point of the family of values $\{v_\Pi\}$.

Proposition

$w(t,\zeta)$ *is a viscosity solution of:*

$$0 = \frac{d}{dt}w(t,\zeta) + \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\langle \zeta, \mathbf{g}(.,\mathbf{x}(.),\mathbf{y}(.))\rangle k(t) + \langle f(\zeta,\mathbf{x},\mathbf{y}), \nabla w(t,\zeta)\rangle] \tag{6}$$

The recursive equation (6) is similar to the one induced by the discretization of a deterministic differential game $\mathscr{G}$ on $\mathbb{R}^+$ defined as follows:

1) the state space is $\Delta(\Omega)$,

2) the action spaces are $\mathbf{X} = X^\Omega$ and $\mathbf{Y} = Y^\Omega$,

3) the dynamics on $\Delta(\Omega) \times \mathbb{R}^+$ is:

$$\dot{\zeta}_t(z) = \sum_{\omega \in \Omega} \zeta_t(\omega) \mathbf{q}(\mathbf{x}(\omega), \mathbf{y}(\omega))[\omega, z]$$

of the form:

$$\dot{\zeta}_t = f(\zeta_t, \mathbf{x}, \mathbf{y})$$

4) the current payoff is given by:

$$\langle \zeta, \mathbf{g}(., \mathbf{x}(.), \mathbf{y}(.)) \rangle = \sum_{\omega \in \Omega} \zeta(\omega) \mathbf{g}(\omega, \mathbf{x}(\omega), \mathbf{y}(\omega)).$$

5) the total evaluation is

$$\int_0^{+\infty} \gamma_t k(t) dt$$

where γ_t is the payoff at time t.

In $\mathscr{G}_\Pi$ the state is deterministic and at each time t_n the players know ζ_{t_n} and choose $\mathbf{x}_n$ (resp. $\mathbf{y}_n$). Let $\mathscr{V}_\Pi(t,\zeta)$ be the associated value.

Proposition

$$\begin{aligned}\mathscr{V}_\Pi(t_{n-1},\zeta_{t_{n-1}}) \quad = \quad & \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\int_{t_{n-1}}^{t_n}\langle\zeta_s,\mathbf{g}(.,\mathbf{x}(.),\mathbf{y}(.))\rangle k(s)ds \\ & +\mathscr{V}_\Pi(t_n,\zeta_{t_n})]\end{aligned} \tag{7}$$

Proposition

The next equation has a unique viscosity solution:

$$0=\frac{d}{dt}U(t,\zeta)+\mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\,[\langle\zeta,\mathbf{g}(.,\mathbf{x}(.),\mathbf{y}(.))\rangle k(t)+\langle f(\zeta,\mathbf{x},\mathbf{y}),\nabla U(t,\zeta)\rangle] \tag{8}$$

Corollary

Both families w_Π and v_Π converge to some w and v with

$$w(t,\zeta) = \sum_{\omega} \zeta(\omega) v(t,\omega).$$

v is a viscosity solution of

$$0 = \frac{d}{dt} v(t,z) + \mathtt{val}_{X\times Y}\{\mathbf{g}(z,x,y)k(t) + \mathbf{q}(x,y)[z,.] \circ v(t,\cdot)\}. \qquad (9)$$

Stationary case

If $k(t) = \rho e^{-\rho t}$, $v(t,z) = e^{-\rho t} \mathbf{v}(z)$ satisfies (9) iff $\mathbf{v}(z)$ satisfies:

$$\rho \ \mathbf{v}_\rho(z) = \mathtt{val}_{X \times Y}[\rho \ g(z,x,y) + \mathbf{q}(x,y)[z,.] \circ \ \mathbf{v}_\rho(.)] \qquad (10)$$

Guo and Hernandez-Lerma (2003), Prieto-Rumeau and Hernandez-Lerma (2012), Neyman (2013), Sorin and Vigeral (2015).

State controlled and not observed: no signals

In the current framework the process Z_t is controlled by both players but not observed.
The moves are observed: we are thus is the symmetric case were the new state variable is $\zeta_t \in \Delta(\Omega)$, the law of Z_t.
Similar framework for differential games in Cardaliaguet and Quincampoix (2008).
Even in the stationary case there is no explicit smooth solution to the basic equation hence a direct approach for proving convergence is not available.

Proposition

The value V_Π satisfies the following recursive equation:

$$V_\Pi(t_{n-1}, \zeta_{t_{n-1}}) \quad = \quad \mathtt{val}_{X\times Y}\, \mathsf{E}_{x,y}[\int_{t_{n-1}}^{t_n} \mathbf{g}(\zeta_s, x, y) k(s) ds + V_\Pi(t_n, \zeta_{t_n})]$$

Proposition

The family of values $\{V_\Pi\}$ *has at least an accumulation point as* $\bar{\delta}$ *goes to 0.*

Proposition

Any accumulation point V *of the family of values* $\{V_\Pi\}$ *is a viscosity solution of:*

$$0 = \frac{d}{dt}V(t,\zeta) + \mathtt{val}_{X\times Y}[\mathbf{g}(\zeta,x,y)\mathbf{k}(t) + \langle \zeta * \mathbf{q}(x,y), \nabla V(t,\zeta)\rangle]. \quad (11)$$

with

$$\zeta * \mu(z) = \sum_{\omega\in\Omega} \zeta(\omega)\mu[\omega,z].$$

The previous computation shows that the limit behavior is the same that the one of the discretization of the differential game with moves X and Y, dynamics on $\Delta(\Omega) \times \mathbb{R}^+$ given by:

$$\dot{\zeta}_t = \zeta_t * \mathbf{q}(x, y).$$

current payoff $\mathbf{g}(\zeta, x, y)$ and evaluation k.

Proposition

Equation (11) has a unique viscosity solution hence the family of values V_Π converge.

Stationary case

In this case one has $V(\zeta,t) = e^{-\rho t}U(\zeta)$ hence (11) becomes

$$\rho U(\zeta) = \mathtt{val}_{X\times Y}[\rho\, \mathbf{g}(\zeta,x,y) + \langle \zeta * \mathbf{q}(x,y), \nabla U(\zeta)\rangle] \qquad (12)$$

Extensions and comments

Incomplete information
Cardaliaguet, Rainer, Rosenberg and Vieille (2015)

General signals

$k \to \infty$

continuous time, Neyman (2012)

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