Asymptotic value in frequency-dependent games
A differential approach

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The class of frequency dependent games

A frequency dependent game is a discrete time repeated game with stage payoffs depending on the choices of the players at that stage and on the relative frequencies whereby all actions have been chosen by the players at previous stages.

Frequency dependent games were introduced by Joosten, Brenner and Witt in 2003. For instance, players' payoffs may change due to learning, habit formation, addiction, or satiation.

The past actions define the state of the game. Both players know the current state, as well as the entire history.

A frequency dependent game can be alternatively seen as a stochastic game with a countable state space.

Asymptotic value in stochastic games

Some of the main questions in the theory of zero-sum stochastic games are related to the asymptotic behavior of the values of these games as players grow more and more patient:

- Does the value of the n-stage game converge as n tends to infinity ?
- Does the value of the λ-discounted game converge as λ tends to 0 ?
- Are the two limits equal ?

Historicity on the differential approach

- The idea of studying the asymptotic value of a discrete time repeated game as the value of a game in continuous time with fixed duration appears in Big Match with lack of information on one side in Sorin 1984.
- A differential game is introduced in Vieille 1996, to study weak approachability for repeated games with vector payoffs.
- The same dynamics occurs in the differential game defined by Laraki 2000, where it is proved existence of the asymptotic value in repeated games with incomplete information on one side (Aumann-Maschler model).

- Sorin 2011 underlines that considering the associated differential game, the same mathematical tools are needed to prove existence of the asymptotic value in both n-stage and λ-discounted games.

- Cardaliaguet et al 2012 achieves a transposition to discrete games of the numerical schemes used to approximate the value function of differential games via viscosity solution arguments, as presented in Barles and Souganidis, 1991.

Model

We introduce a zero-sum repeated game consisting of:

- Two players: P1- the maximizer and P2- the minimizer.
- I and J, the finite action sets of P1 and P2.
- A state space $\mathcal{Z} = \mathbb{N}^{I \times J}$.
- If at time t the state is z and the players choose $i_t \in I$ and $j_t \in J$ respectively then, the transition in $\mathcal{Z}$ is given as follows:

$$z \rightsquigarrow z + e_{i_t j_t}.$$

where e_{ij} is the standard basis vector in $\mathbb{R}^{I \times J}$.

An example on the transition in $\mathcal{Z}$

$I = \{T, B\}$ and $J = \{L, R\}$. If

- at stage $t = 0$, P1 chooses B and P2 chooses L,
- at stage $t = 1$, P1 chooses B and P2 chooses R,

then, the state variable z evolves as it is shown next:

stage $t = 0$ | **stage** $t = 1$ | **stage** $t = 2$

$$z_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad z_1 = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{z_0} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad z_2 = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{z_1} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- $\|z_0\|_1 = 4 + 0$, $\qquad \|z_1\|_1 = 4 + 1$, $\qquad \|z_2\|_1 = 4 + 2$
- and so on...

Model - Payoff function

- Players use mixed strategies. If at time t the state is z_t and the players choose $u_t \in \Delta(I)$ and $v_t \in \Delta(J)$ then, the payoff at this stage that P2 pays to P1 is:

$$\underbrace{g(z_t, u_t, v_t)}_{stage\ payoff} = \underbrace{h(z_t)}_{externality\ payoff} + \underbrace{u_t A v_t}_{current\ payoff},$$

where,

  - $A \in \mathbb{R}^{I \times J}$,
  - $h : \mathcal{Z} \to \mathbb{R}$ defined such that:

$$h(z_t) = \begin{cases} \left\langle H, \frac{z_t}{\|z_t\|_1} \right\rangle, & z_t \neq \mathbf{0} \\ 0, & z_t = \mathbf{0}, \end{cases}$$

where $H \in \mathbb{R}^{I \times J}$.

Model - Strategies

- The set of all histories up to the stage t is denoted by $\mathcal{H}_t$. If $h_t \in \mathcal{H}_t$ then $h_t = (z_0, i_0, j_0, i_1, j_1, \ldots, i_{t-1}, j_{t-1})$.

- A behavioral strategy of P1 is a family of functions $\sigma = (\sigma_t)_t$, where

$$\sigma_t : \mathcal{H}_t \to \Delta(I).$$

  Likewise, a behavioral strategy for P2 is given by $\tau = (\tau_t)_t$.

  The set of behavioral strategies are denoted by Σ and $\mathcal{T}$.

- Given $z_0 \in \mathcal{Z}$, each couple (σ, τ) induces a unique probability distribution $\mathbb{P}^{z_0}_{\sigma,\tau}$ on the set $\mathcal{Z} \times (I \times J)^\infty$ of plays and $\mathbb{E}^{z_0}_{\sigma,\tau}$ stands for the corresponding expectation.

Model - Values of the game

- The total payoff in the game of length N with initial state $z_0 \in \mathcal{Z}$, in which the selected strategies are given by $\sigma = (\sigma_1, ..., \sigma_N)$ and $\tau = (\tau_1, ..., \tau_N)$ respectively, is:

$$G(z_0, \sigma, \tau) = \mathbb{E}_{\sigma,\tau}\left[\frac{1}{N}\sum_{t=0}^{N-1} g(z_t, \sigma_t, \tau_t)\right]$$

- The lower and upper values of the game are defined as follows:

$$\mathbf{V}_N^-(z_0) = \sup_{\sigma\in\Sigma}\inf_{\tau\in T} G(z_0, \sigma, \tau)$$

$$\mathbf{V}_N^-(z_0) = \inf_{\tau\in T}\sup_{\sigma\in\Sigma} G(z_0, \sigma, \tau)$$

Existence of the value in the finite game

Given $n \in \mathbb{N}^*$ and a state $z \in \mathcal{Z}$, the game $\Gamma_n(z)$ has a value, $\mathbf{V}_n(z)$. Moreover, the value of the game satisfies the recursive formula:

$$(n+1)\mathbf{V}_{n+1}(z) = h(z) + \max_{u \in U} \min_{v \in V} \left\{ \sum_{i,j} u_i v_j \Big(a_{ij} + n\mathbf{V}_n(z + e_{ij}) \Big) \right\},$$

- To study the asymptotic behavior of $\mathbf{V}_n$ as the length of the game tends to infinity, we throw bridge accross Shapley and Bellman equations.

The auxilliary game

- The quotient state space: $\mathcal{Q}_N := \left\{ q_N = \frac{z}{N} : z \in \mathcal{Z} \right\}$
- The uniform partition of the time interval $[0,1]$: $\mathcal{I}_N$

- Define $\Psi_N : \mathcal{I}_N \times \mathcal{Q}_N \to \mathbb{R}$ such that,

$$\Psi_N(t_N, q_N) = (1 - t_N)\mathbf{V}_{\underbrace{N(1-t_N)}_{n+1}}(Nq),$$

where $t_N = 1 - \left(\frac{n+1}{N}\right)$.

The auxilliary game

- Previous proposition and definition imply that Ψ_N satisfies the recursive formula:

$$\Psi_N(t_N, q_N) = \frac{1}{N} h(q_N) + \max_{u \in U} \min_{v \in V} \left\{ \sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N \left(t_{N+1}, q_N + \frac{e_{ij}}{N} \right) \right) \right\}$$

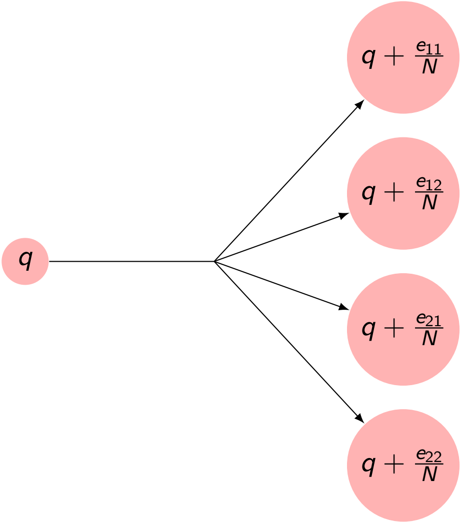
Heuristic derivation of a (PDE)

We assume that $(t_N, q_N) \xrightarrow[N \to \infty]{} (t, q) \in [0,1] \times \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}$ and $\Psi_N(t_N, q_N) \xrightarrow[N \to \infty]{} \Psi(t, q)$. Then, $\Psi : [0,1] \times \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}$ is a bounded, continuous function that satisfies next (PDE) :

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t, q) + h(q) + \max_{u \in U} \min_{v \in V} \left\{ \sum_{ij} u_i v_j \left(a_{ij} + \frac{\partial \Psi}{\partial q_{ij}}(t, q) \right) \right\} = 0, \\ \Psi(1, q) = 0. \end{cases}$$

Example

time t **time t+$\frac{1}{N}$.** $\underbrace{(u, 1-u)}_{P1}, \underbrace{(v, 1-v)}_{P2},$



The continuous time game

The family of differential games $\left(\mathcal{G}(t,q)\right)_{t,q}$

Given $(t,q) \in [0,1] \times \mathbb{R}_+^{I\times J}$, we define $\mathcal{G}(t,q)$ that consists of:

- The time interval of the game, $T = [t,1]$.
- The state space $\mathcal{Q} = \mathbb{R}_+^{I\times J}$.
- The measurable controls $u : [t,1] \to \Delta(I)$, $v : [t,1] \to \Delta(J)$.
- The dynamics in the state space:

$$\begin{cases} \frac{dq}{ds}(s) = u(s) \otimes v(s), & s \in (t,1) \\ q(t) = q_0. \end{cases}$$

- The running payoff is given by $g : \mathbb{R}_+^{I \times J} \times \Delta(I) \times \Delta(J) \to \mathbb{R}$, defined as

$$g(q,u,v) = h(q) + uAv,$$

where,

$$h(q) = \begin{cases} \left\langle H, \frac{q}{\|q\|_1} \right\rangle, & q \neq \mathbf{0} \\ 0, & q = \mathbf{0}. \end{cases}$$

- The total payoff at time $t = 1$ that P2 pays to P1 is:

$$G(q_0,u,v) = \left\langle H, \int_0^1 \frac{q(s)}{\|q(s)\|_1} ds \right\rangle + \left\langle \int_0^1 u(s) \otimes v(s) ds, A \right\rangle$$

where $\|q(s)\|_1 = \|q_0\|_1 + s$.

The continuous time game

Definition

A non anticipative strategy for Player 1 is a function $\alpha : \mathcal{V}_t \to \mathcal{U}_t$, such that for any time $\tilde{t} > t$,

$$\tilde{v}_1(s) = \tilde{v}_2(s) \quad \forall s \in [t, \tilde{t}] \quad \Rightarrow \quad \alpha[\tilde{v}_1(s)] = \alpha[\tilde{v}_2(s)] \quad \forall s \in [t, \tilde{t}].$$

Analogous definition of non anticipative strategies for Player 2.

The lower and upper value functions:

$$\mathbf{W}^-(t, q) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} G(t, q, \alpha[\tilde{v}], \tilde{v})$$

$$\mathbf{W}^+(t, q) := \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} G(t, q, \tilde{u}, \beta[\tilde{u}]).$$

When this information advantage is irrelevant, i.e. both functions coincide, we say that the game $\mathcal{G}(t, q)$ has a value, $\mathbf{W}(t, q)$.

The continuous time game

Definition

The lower and upper hamiltonian functions of the game $\mathcal{G}(t, q)$ $\mathcal{H}^{\pm} : \mathbb{R}_+^{I\times J} \times \mathbb{R}_+^{I\times J} \to \mathbb{R}$, are defined as:

$$\mathcal{H}^{-}(\xi, q) = h(q) + \max_{u\in U} \min_{v\in V} \langle u \otimes v, A + \xi \rangle$$

$$\mathcal{H}^{+}(\xi, q) = h(q) + \min_{v\in V} \max_{u\in U} \langle u \otimes v, A + \xi \rangle,$$

Isaacs condition

The lower and upper hamiltonians verify Isaacs condition i.e.,

$$\mathcal{H}^{-} \equiv \mathcal{H}^{+}$$

The continuous time game

Regularity conditions on dynamics and payoff functions

- The dynamics f is bounded, continuous in all its variables and Lipschitz in q.
- The running payoff g is bounded, continuous in all its variables and Lipschitz in q over $\mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}$.

Results from the theory of differential games

- Let us fix initial state $q_0 \in \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}$. Then, for all $t \in [0,1)$, the differential game with initial state q_0 played over $[t,1]$ has a value, denoted by $\mathbf{W}(t, q_0)$.

- The function $\mathbf{W}(t,q)$ is the unique solution in the space of bounded and continuous functions defined over the set $[0,1] \times \mathbb{R}_+^{I \times J} \setminus \{\mathbf{0}\}$, of the following (HJBI) equation with boundary condition:

$$\begin{cases} \frac{\partial W}{\partial t}(t,q) + \mathcal{H}\left(\frac{\partial W}{\partial q_{ij}}, q\right) = 0, \\ W(1,q) = 0. \end{cases}$$

Remark

Identification of (PDE) and (HJBI).

The discretized differential game

The family of discretized differential games, $\left(\mathcal{G}_n(t_0, q_0)\right)_{t_0, q_0}$

- Given $n \in \mathbb{N}^*$, for all $(t_0, q_0) \in [0,1) \times \mathcal{Q}$, we associate to $\mathcal{G}(t_0, q_0)$ a discrete time game adapted to the uniform subdivision of $[t_0, 1]$ in n intervals $\mathcal{P}_n$, denoted by $\mathcal{G}_n(t_0, q_0)$ starting at time t_0 with initial state q_0 and repeated n times.

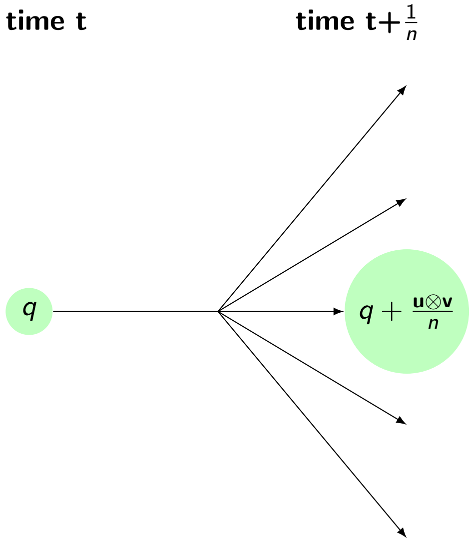
- From Bellman (1957), it follows that the value of that discretized differential game is characterized as follows:

$$W_{\mathcal{P}_n}(t_k, q) = \frac{1}{n} h(q) + \max_{u \in U} \min_{v \in V} \left\{ \left\langle \frac{u \otimes v}{n}, A \right\rangle + W_{\mathcal{P}_n}\left(t_{k+1}, q + \frac{u \otimes v}{n}\right) \right\},$$

where $t_k = \frac{k}{n}$ for $k = 0, \dots, n-1$.

Example

$$\underbrace{(u, 1-u)}_{P1}, \underbrace{(v, 1-v)}_{P2}.$$



Links between the approaches

For all $k \in \{0, ..., n\}$, one may compare Ψ_n and $W_{\mathcal{P}_n}$.

$$\Psi_n(t_k, q) = \frac{1}{n} h(q) + \max_{u \in U} \min_{v \in V} \left\{ \sum_{i,j} u_i v_j \left[\frac{a_{ij}}{n} + \Psi_n \left(t_{k+1}, q + \frac{e_{ij}}{n} \right) \right] \right\}$$

$$W_{\mathcal{P}_n}(t_k, q) = \frac{1}{n} h(q) + \max_{u \in U} \min_{v \in V} \left\{ \sum_{i,j} u_i v_j \frac{a_{ij}}{n} + W_n \left(t_{k+1}, q + \frac{u \otimes v}{n} \right) \right\}$$

Links between the approaches

- Both functions have the same terminal condition, i.e.,

$$\Psi_n(1,q) = W_{\mathcal{P}_n}(1,q) = 0, \quad \forall q \in \mathbb{R}_+^{I \times J}.$$

- Do both functions satisfy the same recursive structure ?

- If next equality holds true,

$$\sum_{i,j} u_i v_j \Psi_n\left(t_{k+1}, q + \frac{e_{ij}}{n}\right) = \Psi_n\left(t_{k+1}, q + \frac{u \otimes v}{n}\right)$$

then, **Yes!**

Properties of the value function

For any $t \in \mathbb{N}$, let us define:

$$\Pi_t := \Big\{ z \in \mathcal{Z} : \|z\|_1 = \underbrace{t}_{\textit{current time}} \Big\}.$$

Proposition

For all $(n, t) \in \mathbb{N}^2$, there exist $k_{n,t} \in \mathbb{R}^{I \times J}$ and $c_{n,t} \in \mathbb{R}$ such that for all $z \in \Pi_t$, one may get

$$n\mathbf{V}_n(z) = \left\langle k_{n,t}, \frac{z}{\|z\|_1} \right\rangle + c_{n,t}.$$

Proof of the property

Assume the result is true for n and we prove it for $n+1$. True for n implies that for all $t \in \mathbb{N}$, there exist $k_{n,t} \in \mathcal{M}^{I \times J}$ and $c_{n,t} \in \mathbb{R}$, such that:

$$n\mathbf{V}_n(z) = \left\langle k_{n,t}, \frac{z}{\|z\|_1} \right\rangle + c_{n,t} \quad , \quad z \in \Pi_t .$$

For $n+1$, for all $z \in \Pi_t$ we get,

$$(n+1)\mathbf{V}_{n+1}(z) = \left\langle H, \frac{z}{\|z\|_1} \right\rangle + \max_{u \in U} \min_{v \in V} \left\{ \sum_{ij} u_i v_j \left(a_{ij} + n \left\langle k_{n,t+1}, \frac{z + e_{ij}}{\|z + e_{ij}\|_1} \right\rangle + n c_{n,t+1} \right) \right\}$$

$$= \left\langle \frac{H}{\|z\|_1} + \frac{n k_{n,t+1}}{\|z\|_1 + 1}, z \right\rangle + \max_{u \in U} \min_{v \in V} \left\{ \sum_{ij} u_i v_j \left(a_{ij} + n \left\langle k_{n,t+1}, \frac{e_{ij}}{\|z\|_1 + 1} \right\rangle + n c_{n,t+1} \right) \right\}$$

Proof of the property

Hence,

$$(n+1)\mathbf{V}_{n+1}(z) = \left\langle k_{n+1,t}, \frac{z}{\|z\|_1} \right\rangle + c_{n+1,t} = \left\langle \frac{k_{n+1,t}}{\|z\|_1}, z \right\rangle + c_{n+1,t}.$$

Then, for all $z \in \Pi_t$, it follows

$$k_{n+1,t} = H + n\frac{t}{t+1}k_{n,t+1}$$

and also,

$$c_{n+1,t} = \max_{u \in U}\min_{v \in V}\left\{\sum_{ij} u_i v_j \left(a_{ij} + n\left\langle k_{n,t+1}\frac{e_{ij}}{t+1}\right\rangle\right)\right\} + nc_{n,t+1},$$

Properties of the value function

Corollary

The general terms of the sequences $\{k_{n,t}\}_{n,t\in\mathbb{N}}$ and $\{c_{n,t}\}_{n,t\in\mathbb{N}}$ are given respectively by:

$$k_{n+1,t} = \frac{H}{n+1}\left(1 + t\sum_{\kappa=t+1}^{n+t}\frac{1}{\kappa}\right)$$

$$c_{n+1,t} = \frac{1}{n+1}\sum_{m=1}^{n}\left[\max_{u\in U}\min_{v\in V}\left\{u\left(A + \frac{H\left(1+(n+t+1-m)\sum_{\kappa=n+t+2-m}^{n+t}\frac{1}{\kappa}\right)}{t+1}\right)v\right\}\right]$$

Coincidence of Ψ_n and $W_{\mathcal{P}_n}$

Given $n \in \mathbb{N}^*$ and $t \in \mathcal{P}_n$, let us define the subset of $\mathcal{Q}_n$:

$$\mathcal{Q}_n(t) = \left\{ q \in \mathcal{Q}_n : \|q\|_1 = t \right\}.$$

Corollary

For all $t \in \mathbb{N}$, the function $\Psi_n(t, \cdot)$ is affine over $\mathcal{Q}_n(t)$.

Proposition

For all $t \in \mathcal{P}_n$ and all $q \in \mathcal{Q}_n(t)$, it holds true:

$$\Psi_n(t, q) = W_{\mathcal{P}_n}(t, q) \, .$$

Existence of the value in the game $\mathcal{G}(0, \mathbf{0})$

Lemma

For all $t \in [0,1]$ and all $(q, \tilde{q}) \in \mathbb{R}^{I \times J} \setminus \{\mathbf{0}\} \times \mathbb{R}^{I \times J}$, it holds true:

$$\left|\mathbf{W}^{\pm}(t,q) - \mathbf{W}^{\pm}(t,\tilde{q})\right| \le 2\Big(\|H\| \Big(\|q\| + \big|\log(\|q\| + t)\big| \Big) + \|A\| \Big) \|q - \tilde{q}\|$$

Proposition

For all $\varepsilon > 0$, there exists $\eta > 0$, such that for all $q, \tilde{q} \in \mathbb{R}^{I \times J} \setminus \{\mathbf{0}\}$ and all $t, \tilde{t} \ge 0$ with $t < \tilde{t}$ and $|q|, |\tilde{q}|, t, \tilde{t} < \eta$, one has

$$\left|\mathbf{W}(t,q) - \mathbf{W}(\tilde{t},\tilde{q})\right| < \varepsilon.$$

Notation. $\lim_{\substack{t \to 0 \\ q \to \mathbf{0}}} \mathbf{W}(t,q) = \ell \in \mathbb{R}$

Existence of the value in the game $\mathcal{G}(0, \mathbf{0})$

Theorem

The differential game starting at time $t = 0$ with initial state $q = \mathbf{0}$ has a value $\mathbf{W}(0, \mathbf{0})$, i.e., $\mathbf{W}^-(0, \mathbf{0}) = \mathbf{W}^+(0, \mathbf{0})$ that is equal to ℓ.

The idea of the proof lies in the consideration of strategies in the game $\mathcal{G}(0, \mathbf{0})$, such that both players play any pure action between times $t = 0$ and $t = \eta$ and then, they play optimally in the game $\mathcal{G}(\eta, q)$, for some $q \in \mathbb{R}^{I \times J} \setminus \{\mathbf{0}\}$. We then show that $\mathbf{W}^-(\mathbf{0})$ and $\mathbf{W}^+(\mathbf{0})$ are both ε-close to ℓ.

Existence of the asymptotic value

Lemma

If $\mathbf{V}_n(\mathbf{0})$ converges to some $\alpha \in \mathbb{R}$ as n tends to infinity, then for all $z \in \mathcal{Z}$, one has

$$\lim_{n \to +\infty} \mathbf{V}_n(z) = \alpha.$$

Theorem

For all $z_0 \in \mathcal{Z}$, the value of the frequency dependent game $\mathbf{V}_n(z_0)$ converges, as n goes to infinity, to the value of its associated differential game played over $[0, 1]$, starting at initial state $\mathbf{0}$, $\mathbf{W}(0, \mathbf{0})$.

Sketch of the proof

- For all $\eta > 0$, there exists $n_0 \in \mathbb{N}^*$, such that for all $n \geq n_0$ one has $\frac{1}{n} < \eta$.

- We consider the game $\mathcal{G}\left(\frac{1}{n}, q_1\right)$, where $q_1 = q\left(\frac{1}{n}\right)$ and by previous Proposition, one may have

$$\left|\mathbf{W}(0,\mathbf{0}) - \mathbf{W}\left(\frac{1}{n}, q_1\right)\right| < \frac{\varepsilon}{3}.$$

- From the theory of differential games, there exists $\delta > 0$, such that for all discretizations $\mathcal{P}$ with mesh $|\mathcal{P}| < \delta$, one has that $W_{\mathcal{P}}$ converges to $\mathbf{W}$ as $|\mathcal{P}| \to 0$. Thus, there exists $n_1 := \lfloor \frac{1}{\delta} \rfloor + 1$, such that for all $n \geq n_1$ one has:

$$\left|W_{\mathcal{P}_n}\left(\frac{1}{n}, q_1\right) - \mathbf{W}\left(\frac{1}{n}, q_1\right)\right| < \frac{\varepsilon}{3}$$

- We have: $W_{\mathcal{P}_n}\left(\frac{1}{n}, q_1\right) = \Psi_n\left(\frac{1}{n}, q_1\right)$

- There exists $n_2 := \left\lfloor \frac{3\|A\|_\infty}{\varepsilon} \right\rfloor + 1$ such that for all $n \geq n_2$ one has:

$$\left|\Psi_n(0, \mathbf{0}) - \Psi_n\left(\frac{1}{n}, q_1\right)\right| < \frac{\varepsilon}{3}$$

- Hence, for all $n \geq \max\{n_0, n_1, n_2\}$, we get

$$|\Psi_n(0, \mathbf{0}) - \mathbf{W}(0, \mathbf{0})| < \varepsilon$$

- Recall that $\Psi_n(t_n, q_n) = (1 - t_n)\mathbf{V}_{n(1-t_n)}(nq_n)$.

Extensions

1. Given $\lambda \in (0, 1]$ and $z_0 \in \mathcal{Z}$, consider the discounted game $\Gamma_\lambda(z_0)$. The total payoff of P1 is:

$$G(z_0, \sigma, \tau) = \mathbb{E}_{\sigma,\tau}\left[\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} g(z_t, i_t, j_t)\right].$$

   (?) Do we have as λ goes to zero,

$$\mathbf{V}_\lambda(z_0) \to \mathbf{W}(0, \mathbf{0})$$

2. What if the payoff function in $\Gamma_N(z_0)$ is non-additive ?

Merci pour votre attention !