Approachability of Convex Sets in “Some” Absorbing Games

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Joint work with

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- He proved that a necessary and sufficient condition for a convex set $\mathcal{C}$ to be approachable is:

$$\forall y \in \Delta(I)\ \exists x \in \Delta(J) : \ g(x, y) \in \mathcal{C} \iff \max_{y \in \Delta(J)} \min_{x \in \Delta(I)} d_{\mathcal{C}}(g(x, y)) = 0$$

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  Vieille, [Hart & Mas-Colell], Spinat, Lehrer, Dawid, Renault & Tomala
  [As Soulaimani, Quincampoix & Sorin], Perchet, [Lehrer & Solan]
  Rakhlin, [Sridharan & Tewari], [Perchet & Quincampoix], Lovo, Horner & Tomala
  [Foster & Vohra], [Fudenberg & Levine], [Sandroni, Smorodinsky & Vohra]
  [Hart & Mas-Colell], [Cesa-Bianchi & Lugosi], [Benaim, Hofbauer & Sorin]

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Quitting Games

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Sets of actions:

Pure actions of player 1 (the decision maker): $\mathbf{I} = \mathcal{I} \times \mathcal{I}^*$
Pure actions of player 2 (nature or advisory): $\mathbf{J} = \mathcal{J} \times \mathcal{J}^*$.
Mixed actions of P1 $\mathbf{x} \in \Delta(\mathcal{I} \times \mathcal{I}^*)$, $x \in \Delta(\mathcal{I})$, $x^* \in \Delta(\mathcal{I}^*)$,
Mixed actions of P2 $\mathbf{y} \in \Delta(\mathcal{J} \times \mathcal{J}^*)$, $y \in \Delta(\mathcal{I})$, $y^* \in \Delta(\mathcal{I}^*)$.
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Restrictions

If $\mathcal{J}^* = \emptyset$ then the game is a *Big-match of type I*.
If $\mathcal{I}^* = \emptyset$ then the game is a *Big-match of type II*.

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- Player 1 wants to approach the set $\mathcal{C}$, player 2 wants to avoid $\mathcal{C}$.

Approachability Notions Studied

Uniform Approachability

$\forall \varepsilon > 0$, player 1 has a strategy σ such that after some stage $T \in \mathbb{N}$, $\overline{g}_T = \mathbb{E}_{\sigma,\tau}[\frac{1}{T}\sum_{t=1}^{T} g_t]$ is ε-close to $\mathcal{C}$, no matter the strategy τ of player 2.

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Weak Approachability

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We will study the following stronger notion:

$\forall \varepsilon > 0$, $\exists \theta_\varepsilon > 0$ s.t. $\forall \theta = \{\theta_s\}_{s \in \mathbb{N}^*} \in \Delta(\mathbb{N}^*)$ satisfying $\|\theta\|_2 \leq \theta_\varepsilon$, $\exists \sigma$, $\forall \tau$ $d_{\mathcal{C}}\left(\mathbb{E}_{\sigma,\tau}[\sum_{t=1}^{\infty} \theta_t g_t]\right) \leq \varepsilon$.

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Blackwell condition holds:

$$\forall \mathbf{y} = qL + (1-q)R, \exists \mathbf{x} = (1-q)T + qB : g(\mathbf{x}, \mathbf{y}) = 0$$

Notations

- g is extended multi-linearly to the set of measures on $\mathcal{M}(\mathbf{I})$ and $\mathcal{M}(\mathbf{J})$:

$$g(\alpha, \beta) = \sum_{i \in \mathbf{I}, j \in \mathbf{J}} \alpha_i \beta_j g(i, j).$$

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- We also extend the probability of absorption and the expected absorption payoffs:

$$p^\star(\alpha, \beta) = \sum_{i \in \mathbf{I}, j \in \mathbf{J}} \alpha_i \beta_j - \sum_{i \notin \mathcal{I}^\star} \sum_{j \notin \mathcal{J}^\star} \alpha_i \beta_j$$

and

$$g^\star(\alpha, \beta) = g(\alpha, \beta) - \sum_{i \notin \mathcal{I}^\star} \sum_{j \notin \mathcal{J}^\star} \alpha_i \beta_j g(i, j).$$

The Conditions

- Sufficient condition : SC

$$\max_{\mathbf{y}\in\Delta(\mathbf{J})} \min_{\mathbf{x}\in\Delta(\mathbf{I})} \inf_{\alpha\in\mathcal{M}(\mathbf{I})} \sup_{\beta\in\mathcal{M}(\mathbf{J})} d_C\left(\frac{g(\mathbf{x},\mathbf{y}) + g^\star(\alpha,\mathbf{y}) + g^\star(\mathbf{x},\beta)}{1 + p^\star(\alpha,\mathbf{y}) + p^\star(\mathbf{x},\beta)}\right) = 0 \quad (1)$$

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- Non-necessary, non-sufficient condition:

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- Necessary condition: NC

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Main Results for Weak Approachability

Theorem

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NC (condition 3) is necessary for W-approachability.
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Lemma

Condition SC is equivalent to

(1) $\exists (x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ *such that*

$$g(x_0^*, j) \in \mathcal{C}, \ \forall j \in \mathcal{J}$$
$$\text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

or
(2) $\forall \varepsilon, \ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ *such that:*

$$g((1-\gamma)x + \gamma x^*, y) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

Proof of SC: Part 1

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- By condition SC, if the game is absorbed, the payoff is necessarily in $\mathcal{C}$.
- Consequently,

$$d\left(\mathbb{E}\left[\overline{g}_\theta\right], \mathcal{C}\right) \leq \sum_{s=1}^{\infty}(1-\gamma_0)^s \theta_s M \leq \frac{1-\gamma_0}{\sqrt{2\gamma_0 - \gamma_0^2}} \|\theta\|_2 M$$

Proof of SC: Part 2

Suppose SC is:

$\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ such that:

$$(1-\gamma) g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1)$$
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- By SC, for each $y[k]$, we may associate $(x[k], x^*[k], \gamma[k])$.
- The strategy of player 1 at stage τ (history dependent) is defined as:

$$\gamma_\tau[k_\tau] x^*[k_\tau] + (1 - \gamma_\tau[k_\tau]) x[k]$$

where:

$$\gamma_\tau[k_\tau] := \frac{\gamma[k_\tau]\theta_\tau}{(1 - \gamma[k_\tau]) \sum_{s=\tau}^{\infty} \theta_s + \gamma[k_\tau]\theta_\tau}$$

Necessary Condition

Condition NC (condition 3)

$$\max_{\mathbf{y}\in\Delta(\mathbf{J})} \sup_{\beta\in\mathcal{M}(\mathbf{J})} \min_{\mathbf{x}\in\Delta(\mathbf{I})} \inf_{\alpha\in\mathcal{M}(\mathbf{I})} d_{\mathcal{C}}\left(\frac{g(\mathbf{x},\mathbf{y})+g^{\star}(\alpha,\mathbf{y})+g^{\star}(\mathbf{x},\beta)}{1+p^{\star}(\alpha,\mathbf{y})+p^{\star}(\mathbf{x},\beta)}\right)=0$$

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Theorem

NC is necessary for weak approachability in generalized quitting games.

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If not, player 2 just play at every stage y perturbed by β. This allows him to exclude $\mathcal{C}$.

Weak Approachability in Big Match Type 1

Lemma

In Big-Match of type I, SC and NC are equivalent to Blackwell condition:

$$\forall \mathbf{y} \in \Delta(\mathbf{J}), \exists \mathbf{x} \in \Delta(\mathbf{I}), g(\mathbf{x}, \mathbf{y}) \in \mathcal{C}$$

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- Denote by q^* the probability, that play eventually absorbs. Since

$$u(\sigma,\tau) = \frac{1}{2}q^* - \frac{1}{2}(1-q^*) = q^* - \frac{1}{2},$$

we have

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- Take t large so that the proba q_t that play absorbs before t is at least $\frac{3}{10}$.
- Let τ' the strategy $(\frac{1}{2}, \frac{1}{2})$ at all periods before period t and L after. Then

$$u(\sigma,\tau') \geq \frac{1}{2}q_t \geq \frac{3}{20} > \frac{1}{10},$$

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The set $\mathcal{C} = \{(x,y) : x \geq \frac{3}{8}, y \geq \frac{3}{8}\}$ is neither uniformly approchable nor uniformly excludable.

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Lemma

In Big-Match games of type II, SC (condition 1) is equivalent to

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If $y \in \Delta(\mathcal{J})$ is predicted, P1 plays $x \in \Delta(\mathcal{I})$. And this strategy must remain "good" even if player 2 decides to quit the game.

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If $\mathcal{C}$ is weakly approchable, $\exists$ a measurable mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for almost every $t \in [0,1]$,

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- We tend N to infinity and ε to zero.

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- Now we divide each time interval of length $1/N$ on a large block of length L in which player 1 plays an i.i.d strategies $\xi(s)$.
- By the law of large numbers, on the block L, the average payoff if player 2 plays always R is $g_R(\xi(s))$.

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- Find a C^1 function ξ (where $\xi(s)$ = proba of T at time s) s.t. $\forall t$:

$$\int_0^t (\xi(s)p - (1-\xi(s))ds + (1-t)\xi(t) = 0,$$

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Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ *a continuous mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

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Without absorption, this is Vieille's differential game characterization for W-approachability.

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Thanks to the organizers. It is impossible to do a better conference!