Learning in Mean Field Games: The Fictitious Play

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1. Introduction to Mean Field Games (MFG)
   - Model
   - Mean Field Game Equilibrium

2. Fictitious Play in Mean Field Game
   - Potential Mean Field Game
   - Second Order MFG
   - First Order MFG
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- M. Huang, R.P. Malhamè and P.E. Caines:
  - *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle.* Communication in information and systems (2006). Vol. 6, No. 3, pp. 221-252.

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$$\mathrm{d}x_t = \alpha_t \mathrm{d}t + \sqrt{2\sigma}\mathrm{d}B_t$$

  where B_t is a Brownian motion adapted to some filtration $(\mathcal{F}_t)_{t\in[0,T]}$. In stochastic case ($\sigma \neq 0$) we suppose α_t is adapted to $\mathcal{F}_t$.

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- The case $\sigma = 0(1)$ is called *First(Second) Order* MFG.

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- Best reply to the rest of population:
$$\mathrm{argmin}_{\alpha\in(\mathcal{F}_t)_{t\in[0,T]}} \mathcal{J}(\alpha, (m_t)_{t\in[0,T]})$$

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$$\inf_{x_s = x, \mathrm{d}x_t = \alpha_t \mathrm{d}t + \sigma \mathrm{d}B_t} \mathbb{E} \left(\int_s^T \left(L(x_t, \alpha_t) + f(x_t, m_t) \right) dt + g(x_T, m_T) \right).$$

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- HJB Equation:

$$-\partial_t u - \sigma \Delta u + H(x, \nabla u) = f(x, m_t)$$

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where $H(x, p) = -\inf_q \langle p, q \rangle + L(x, q)$.

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- $\alpha(t,x) = -D_p H(x, \nabla u(t,x))$ the best reply in feedback form.
- The key point is that the agent assumes $m_t \in \mathcal{P}(X), t \in [0,T]$ as given.

The Fokker-Planck

If every one choose their best replies:

$$\alpha(t,x) = -D_pH(x, \nabla u(t,x))$$

then the distribution evolves by Fokker-Planck (or continuity) Equation:

$$\partial_t m - \sigma\Delta m + \mathrm{div}(\alpha(t,x)m) = 0$$

$$m(0) = m_0.$$

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$$(m_t^B)_{t\in[0,T]} \Rightarrow^{HJB} \quad (\alpha_t)_{t\in[0,T]} \Rightarrow^{FP} \quad (m_t^R)_{t\in[0,T]}$$

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Mean Field Game Equilibrium is represented by the solution of the following coupled equations:

$$\begin{cases} (i) & -\partial_t u - \sigma \Delta u + H(x, \nabla u) = f(x, m(t)) \text{ in } (0,T) \times \mathbb{R}^d \\ (ii) & \partial_t m - \sigma \Delta m - \operatorname{div}(m D_p H(x, \nabla u)) = 0 \text{ in } (0,T) \times \mathbb{R}^d \\ (iii) & m(0) = m_0,\ u(x,T) = g(x, m(T)) \text{ in } \mathbb{R}^d \end{cases}$$

Theorem (P.L. Lions, J.M. Lasry)

Suppose the following $(*)$ conditions hold:

- m_0 *has a smooth density.*
- $H \in \mathcal{C}^2(\mathbb{T}^d \times \mathbb{R}^d), \exists C > 0 : C^{-1} I_d \leq D_{pp} H(x,p) \leq C I_d,$
- $\langle D_x H(x,p), p \rangle \geq -C(1 + \|p\|^2),$
- $f : m \to f(\cdot, m)$ *is Lipchitz from* $\mathcal{P}(\mathbb{T}^d)$ *to* $\mathcal{C}^2(\mathbb{T}^d)$,
- $g : m \to g(\cdot, m)$ *is Lipchitz from* $\mathcal{P}(\mathbb{T}^d)$ *to* $\mathcal{C}^3(\mathbb{T}^d)$,
- $\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \|f(\cdot, m)\|_{\mathcal{C}^2} + \|g(\cdot, m)\|_{\mathcal{C}^3} < +\infty.$

Then the there exist (u, m) which satisfy MFG Equilibrium equation in strong(weak) sense in case $\sigma \neq 0 (= 0)$. In addition, if f, g are monotone, then the equilibrium is unique.

We call $h : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ monotone if for every distinct $m, m' \in \mathcal{P}(\mathbb{T}^d)$:

$$\int_{\mathbb{T}^d} \big(h(x,m) - h(x,m')\big)\, \mathrm{d}(m - m')(x) > 0$$

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How the people react with belief $m_t = \bar{m}_t^{(k)}$:

$$\mathcal{J}(\alpha, (\bar{m}_t^{(k)})_{t \in [0,T]}) = \mathbb{E}\left(\int_0^T \left(L(x_t, \alpha_t) + f(x_t, \bar{m}_t^{(k)}) \right) dt + g(x_T, \bar{m}_T^{(k)}) \right)$$

$$-\partial_t u^{(k+1)} - \sigma \Delta u^{(k+1)} + H(x, \nabla u^{(k+1)}) = f(x, \bar{m}_t^{(k)})$$
$$\alpha^{(k+1)}(t, x) = -D_p H(x, \nabla u^{(k+1)}(t, x))$$

The Fokker-Planck equation tells us how the real distribution evolves.

$$\partial_t m^{(k+1)} - \sigma \Delta m^{(k+1)} + \mathrm{div}(\alpha^{(k+1)}(t,x) m^{(k+1)}) = 0, \quad m(0) = m_0$$

And the agents adjust their belief by choosing $m^{(k+1)}$ the solution of precedent Fokker-Planck.

Fictitious play define by induction the sequence $\{(u^k, m^k)\}_{k\in\mathbb{N}}$ as follows: Choose $m^{(0)}$ arbitrary, then for $k = 1, 2, \dots$:

$$\begin{cases} (i) & -\partial_t u^{(k+1)} - \sigma\Delta u^{(k+1)} + H(x, \nabla u^{(k+1)}) = f(x, \bar{m}^{(k)}(t)) \\ (ii) & \partial_t m^{(k+1)} - \sigma\Delta m^{(k+1)} - \mathrm{div}(m^{(k+1)} D_p H(x, \nabla u^{(k+1)})) = 0 \\ (iii) & m^{(k+1)}(0) = m_0,\ u^{(k+1)}(x, T) = g(x, \bar{m}^{(k)}(T)) \end{cases}$$

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- Potential Games
- How can we define the Potential Mean Field Game ?

Potential Mean Field Game

- We call a Mean Field Game as a Potential Mean Field Game, if there exist $F, G : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that:

$$\lim_{s \to 0} \frac{F(m + s(m' - m)) - F(m)}{s} = \int_{\mathbb{T}^d} f(x, m) \mathrm{d}(m' - m)(x)$$

$$\lim_{s \to 0} \frac{G(m + s(m' - m)) - G(m)}{s} = \int_{\mathbb{T}^d} g(x, m) \mathrm{d}(m' - m)(x)$$

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- For example the following quite general case is indeed a Potential MFG:

$$f(x, m) = \int_{\mathbb{T}^d} \phi(x, y) \, \mathrm{d}m(y), \quad g(x, m) = \int_{\mathbb{T}^d} \psi(x, y) \, \mathrm{d}m(y)$$

if ϕ, ψ are symmetric.

Theorem (P. Cardaliaguet , S. Hadikhanloo)

Suppose the conditions $(*)$ *hold. If the MFG is Potential, then every accumulation point of the precompact sequence* $\{(u^k, m^k)\}_{k\in\mathbb{N}}$ *which is constructed by a Fictitious Play, is indeed an equilibrium. Hence, if the equilibrium is unique then the later converge to the unique equilibrium.*

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Suppose the conditions $(*)$ hold. If the MFG is Potential, then every accumulation point of the precompact sequence $\{(u^k, m^k)\}_{k\in\mathbb{N}}$ which is constructed by a Fictitious Play, is indeed an equilibrium. Hence, if the equilibrium is unique then the later converge to the unique equilibrium.

Because of the existence of regularity conditions in case $\sigma \neq 0$, the proof framework is different for Second and First order MFG.

Second Order MFG ($\sigma = 1$)

- Suppose $\{(u^{(k)}, m^{(k)})\}_{k\in\mathbb{N}}$ is constructed by a Fictitious Play. Define for every $k \in \mathbb{N}$:

$$w_t^{(k)}(x) = m_t^{(k)}(x)\alpha^{(k)}(t,x) = -m_t^{(k)}(x) D_p H(x, \nabla u^{(k)}(t,x))$$

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- For any pair $(m, w) \in \mathcal{C}([0,T] \times \mathbb{T}^d, \mathbb{R}) \times \mathcal{C}([0,T], \mathbb{R}^d)$ which satisfies:

$$\partial_t m - \sigma \Delta m + \mathrm{div}(w) = 0,$$

define the potential:

$$\Phi(m, w) =$$

$$\int_0^T \int_{\mathbb{T}^d} m_t(x) H^*(x, -w_t(x)/m_t(x)) \mathrm{d}t\mathrm{d}x + \int_0^T F(m_t)\mathrm{d}t + G(m_T)$$

where $H^*(x,q) = \sup_p \langle q, p\rangle - H(x,p)$.

- One can show that $\Phi(\bar{m}^{(k)}, \bar{w}^{(k)})$ is (almost) decreasing i.e. there exists $C > 0$ such that:

$$\Phi(\bar{m}^{(k+1)}, \bar{w}^{(k+1)}) - \Phi(\bar{m}^{(k)}, \bar{w}^{(k)}) \leq -\frac{a_k}{k} + \frac{C}{k^2},$$

  where $a_k = \int_0^T \int_{\mathbb{T}^d} \bar{m}^{(k)} \|\bar{w}^{(k)}/\bar{m}^{(k)} - w^{(k)}/m^{(k)}\|^2 \geq 0$.

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  where $a_k = \int_0^T \int_{\mathbb{T}^d} \bar{m}^{(k)} \|\bar{w}^{(k)}/\bar{m}^{(k)} - w^{(k)}/m^{(k)}\|^2 \geq 0.$
- Since $\Phi(\bar{m}^{(k)}, \bar{w}^{(k)}),\ k \in \mathbb{N}$ is bounded below we have:

$$\sum_{k\in\mathbb{N}} \frac{a_k}{k} < +\infty \quad \Rightarrow \quad \lim_{k\to\infty} \frac{a_1 + a_2 + \cdots + a_k}{k} = 0.$$

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- There is $C > 0$ so that $|a_{k+1} - a_k| < \frac{C}{k}$ which proves:

$$\lim_{k \to \infty} a_k = 0 \Rightarrow \lim_{k \to \infty} \bar{w}^{(k)}/\bar{m}^{(k)} - w^{(k)}/m^{(k)} =^{PW} 0.$$

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  where $a_k = \int_0^T \int_{\mathbb{T}^d} \bar{m}^{(k)} \|\bar{w}^{(k)}/\bar{m}^{(k)} - w^{(k)}/m^{(k)}\|^2 \geq 0$.
- Since $\Phi(\bar{m}^{(k)}, \bar{w}^{(k)}),\ k \in \mathbb{N}$ is bounded below we have:

$$\sum_{k\in\mathbb{N}} \frac{a_k}{k} < +\infty \quad \Rightarrow \quad \lim_{k\to\infty} \frac{a_1 + a_2 + \cdots + a_k}{k} = 0.$$

- There is $C > 0$ so that $|a_{k+1} - a_k| < \frac{C}{k}$ which proves:

$$\lim_{k\to\infty} a_k = 0 \Rightarrow \lim_{k\to\infty} \bar{w}^{(k)}/\bar{m}^{(k)} - w^{(k)}/m^{(k)} =^{PW} 0.$$

- Any accumulation point of the precompact set:

$$k \in \mathbb{N}, \quad (u^{(k+1)}, m^{(k)}, \bar{m}^{(k)}, \bar{w}^{(k+1)})$$

  is of the form $(u, m, m, -mD_pH(\nabla u))$ where (u, m) is a MFG equilibrium.

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- Set the real distribution made by this belief:

$$\theta^{k+1} = \Psi^{k+1} \sharp m_0.$$

- For every $\eta \in \mathcal{P}(\Gamma)$ define the potential:

$$\Phi(\eta) = \int_\Gamma \int_0^T \frac{1}{2}\|\dot{\gamma}_t\| \mathrm{d}t \mathrm{d}\eta(\gamma) + \int_0^T F(e_t \sharp \eta) \mathrm{d}t + G(e_T \sharp \eta)$$

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- One can show that $\{\Phi(\eta^k)\}_{k\in\mathbb{N}}$ is (almost) decreasing:

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- Then $a_k \to 0$ which yields our result: for any cluster point $\bar{\eta} \in \mathcal{P}_0(\Gamma)$:

$$\int_\Gamma \mathcal{J}(\gamma, \bar{\eta})\, \mathrm{d}\bar{\eta} = \inf_{\theta \in \mathcal{P}(\Gamma), e_0\sharp\theta = m_0} \int_\Gamma \mathcal{J}(\gamma, \bar{\eta})\, \mathrm{d}\theta$$

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- Then his cost will be:

$$\mathcal{J}(\alpha^i, x^{-i}) = \int_0^T \left(L(\alpha_t^i, x_t^i) + f(x_t^i, m_{t,N}) \right) \mathrm{d}t + g(x_T^i, m_{T,N}).$$

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- Here the question is: if $N \to \infty$ does the distribution $\lim_{N\to\infty} \bar{\eta}_N$ exist and if yes then is it equal to MFG equilibrium ?

Theorem (P. Cardaliaguet , S. Hadikhanloo)

Suppose the conditions $(*)$ hold and there is a unique First Order MFG equilibrium. Consider for every $N \in \mathbb{N}$ the initial points $X_N^1, X_N^2, \ldots, X_N^N \in \mathbb{T}^d$ so that:

$$m_{0,N} \to m_0, \quad N \to \infty.$$

Then for every $\epsilon > 0$ a Fictitious play of a First Order MFG with $N-$players can reach to a $\epsilon-$neighborhood of equilibrium whenever N is enough large.