A demographic prisonner's dilemma

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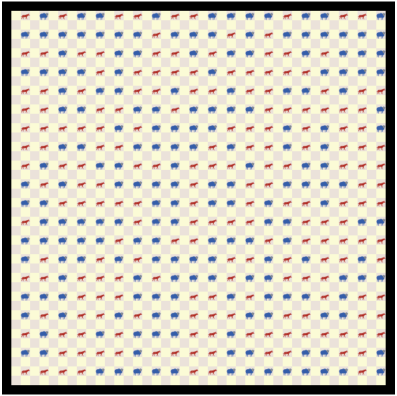
5 janvier 2016
GEL

Spatial prisoner's dilemma

Model [Nowak & May '93]

Players play with their neighbours. They evolve adopting the action of the neighbour with the highest payoff at the previous round.



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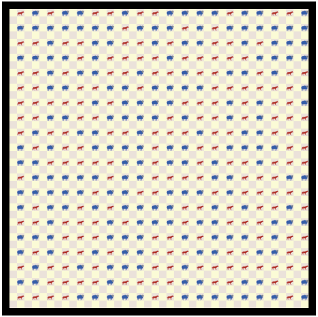
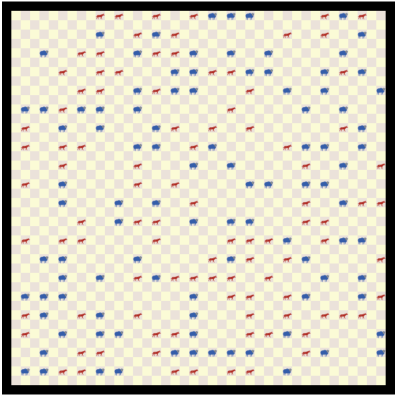


Biological Hypothesis

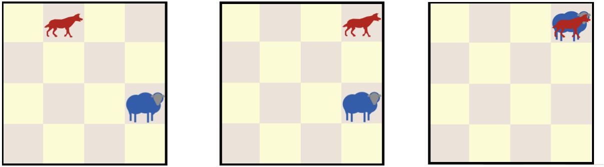
Rate of reproduction $\gg$ rate of death

Without this hypothesis : To the demographic prisoner's dilemma



The players can move on the torus

Move of the player



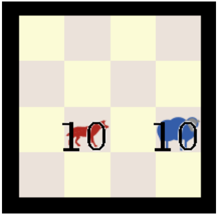
Movement

In this work, players move following continuous independent symmetric simple random walks.

Evolution

Wealth

Each player carries a wealth w.
If $w = 0$, the player die.



Demographic prisoner's dilemma [Epstein '98]

Game and Effect

The game is a prisoner's dilemma of average payoff :

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (R,R) & (-S,T) \\ (T,-S) & (-P,-P) \end{pmatrix} \end{array} .$$

with $T > R > 0$ and $P > S > 0$.
Every player has a unique action :

- the blue players only cooperate,
- the red players only defect.

Spatial condition

Only particles on the same site can play together.

Data

Data

- ρ : density of the defectors, β : density of the cooperators,
- N : number of players,
- q_0 : initial amount of wealth,
- λ_d : move speed,
- λ_g : game speed.
- m : size of the torus $D = (\mathbb{Z}/m\mathbb{Z})^2$

We denote by *Red* the defectors and *Blue* the cooperators.

Definition

A player $(\chi_t)_t$ is a process taking value in $D \times \{Red, Blue\} \times \mathbb{N}$.

Initialisation

Initialisation

1. Drawing N independent random variables equal to *Red* with probability $\frac{\rho}{(\beta+\rho)}$ and *Blue* with probability $\frac{\beta}{(\beta+\rho)}$. This will give the action of the player.
2. Drawing N independent r.v. (independently from the colors) uniform on D.
3. Putting on each site sampled a player with second component the color sampled and third component equal to q_0.

Poisson Processes

Independence

All the Poisson processes will be independent with everything (and between them).

About the movement

Each particle has a Poisson process of parameter λ_m attached. When we have a realization of this Poisson process, the particle will move accordingly to a simple symmetric random walk. All the moves of the particles are independent.

About the game

Each pair of (distinct) particles has a Poisson process of parameter λ_g attached. When we have a realization of this Poisson process, we check if the two particles are on the same site. If so, a prisoner's dilemma occurs between the two players. The wealth of the two players change according the payoff of the game.

Payoff matrices

The prisoner's dilemma payoff matrix is (with $T > R > 0$ and $P > S > 0$) :

$$\begin{pmatrix} (R, R) & (-S, T) \\ (T, -S) & (-P, -P) \end{pmatrix}$$

Remark

With this payoff matrix, two red particles can kill one another.

Payoff matrices

The payoff matrices are with $T > R > 0$ and $P > S > 0$:

$$\begin{pmatrix} (R, R) & (-S, T) \\ (T, -S) & \frac{1}{2}(-2P, 0) + \frac{1}{2}(0, -2P) \end{pmatrix}$$

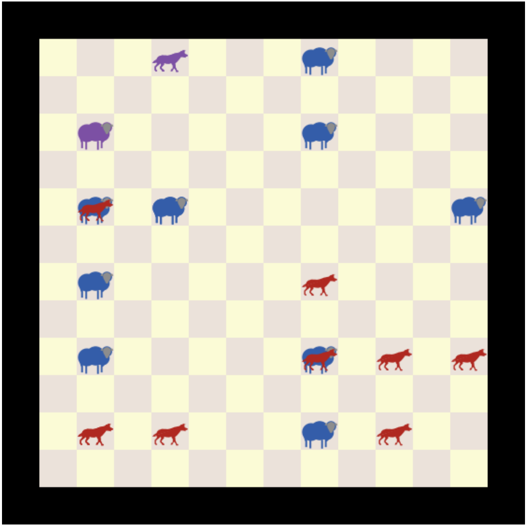
Remark

With this new payoff matrix, two red particles cannot kill one another.

Definition

A configuration σ is an element of $(D \times \{Red, Blue\} \times \mathbb{N})^N$.
A particle system $(\sigma_t)_t$ is a process taking values in the space of configurations.

Simulation on Netlogo



click here

Results

Payoff Matrix

The payoff matrices are with $T > R > 0$ and $P > S > 0$:

$$\begin{pmatrix} (R,R) & (-S,T) \\ (T,-S) & \frac{1}{2}(-2P,0)+\frac{1}{2}(0,-2P) \end{pmatrix}$$

Theorem

There exists a constant $\mu > 0$ depending only on $\lambda_b, \lambda_m, \lambda_g, N$ and m such that if :

$$\mu R < S$$

then for each initial configuration :
The cooperators will die almost surely.

Payoff Matrix

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Theorem

There exists a constant $\mu_1 > 0$ depending only on $\lambda_b, \lambda_m, \lambda_g$, N and m such that if :

$$\mu_1 S < R$$

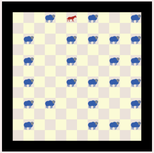
then for each initial configuration :

$$\mathbb{P}(\{\textit{the cooperators live ad vitam eternam}\}) > 0$$

Sketch of the first theorem proof

Context

Only one red particle and blue particles.



Objective

Let W_t^{tot} the sum of the wealth of the blue particles at time t.
Showing that :

$$W_t^{tot} \underset{t \to +\infty}{\longrightarrow} 0 \qquad \text{a.s.}$$

Usefull notation for the proof

$\tau(\sigma) = \inf\{n \geq 0,\ t_n$ is the realization of a game Poisson process between a blue player and the red player on the same site $\}$. $\tau(\sigma)$ doesn't depend on the wealth of the players.

$p(\sigma) =$ probability that this game happens in less than $2m + 1$ realizations of Poisson processes going from a configuration σ. $p(\sigma)$ doesn't depend on the wealth of the players.

$p = \min_{\sigma} p(\sigma)$

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sketch of the proof of the second theorem

Theorem

There exists a constant $\mu_1 > 0$ depending only on $\lambda_b, \lambda_m, \lambda_g, N$ and m such that if :

$$\mu_1 S < R$$

then for each initial configuration :

$$\mathbb{P}(\{\textit{the cooperators live ad vitam eternam}\}) > 0$$

Idea

The main idea is to consider a ghost system, such that in it, the player don't die but can have negative wealth.
In this system, with the first theorem and with other hypothesis $W_t^{tot} \to +\infty$ a.s. when $t \to +\infty$.

Mean Field

Mean field assumption

- There is an infinite number of players.
- All the particles have independent laws.
- All the red particle wealths have the same laws (law of a process $(R_t)_t$)
- All the blue particles wealths have the same laws (law of a process $(B_t)_t$).

Spatial approximation

At time $t > 0$:
Particles play against a red particle with probability $\rho\mathbb{P}(R_t > 0)$.
Particles play against a blue particle with probability $\beta\mathbb{P}(B_t > 0)$.

We call the induced stochastic process : $(\sigma_t^{mf})_t$.

Mean Field Result

Theorem

Let $\eta > 1$ *satisfying*

$$q_0 - \frac{\eta^2 \lambda_g (C^2 + S^2)}{4(\beta(1 - \frac{1}{\eta^2})C - \rho S)} > 0$$

called the starter condition and also satisfying :

$$\beta C - \rho S > C/\eta^2.$$

Then we have : $\forall t > 0$

$$\mathbb{P}(B_t > 0) \geq 1 - \frac{1}{\eta^2} > 0$$

Example

For $q_0 = 50, \lambda_g = 1, S = 2, R = 1, \beta = 0.6, \rho = 0.2$ we have a density of blue player always higher than 52%.

Intermediate Model

Model

- Finite number of players.
- All the players are on the same site.
- Spatial condition replaced by :
  Game cancelled with probability : $1 - (1/m)^2$

We call the induced stochastic process : $(\hat{\sigma}_t)_t$

Intermediate model : Result and Goals

Theorem

We have the following convergence in law when the move speed goes to $+\infty$ *:*

$$\forall t > 0, \qquad \mathcal{L}(W_t) \underset{\lambda_d \to +\infty}{\longrightarrow} \mathcal{L}(\hat{W}_t).$$

Short term goal

Proving the pathwise convergence : for $T > 0$

$$\mathcal{L}(\sigma_{[0,T]}) \underset{\lambda_d \to +\infty}{\longrightarrow} \mathcal{L}(\hat{\sigma}_{[0,T]}).$$

Goal : Propagation of chaos

Proving the pathwise convergence : for $T > 0$

$$\mathcal{L}(\hat{\sigma}_{[0,T]}) \underset{\substack{\lambda_d \to +\infty \\ N \to +\infty}}{\longrightarrow} \mathcal{L}(\sigma^{mf}_{[0,T]}).$$

Other way of moving : Instinctive move with curiosity

Way of moving

Let $p > 0$,

- With probability p : the particle move randomly
- With probability $1 - p$: the particle move instinctively *i.e.*
  If its last encounter is with a blue particle it stays else it moves.

Theorem

The two first theorem (almost sure extinction and ad vitam eternam survival) hold.

Instinctive moving : Simulation

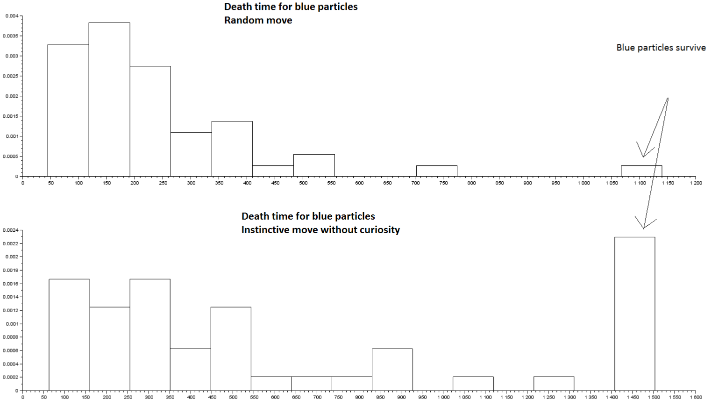

FIGURE: Comparison survival with instinctive moving and with random moving

Extension : Birth

Data

- w_c : necessary amount of wealth to give birth,
- w_0 : initial amount of wealth of the babies.

Poisson process

Each particle has a Poisson process of parameter λ_b. When we have a realization of this Poisson process, we check if the wealth of the player is greater than w_c. If so, a player of the same color appears on the same site of the first player. The child will have a wealth equal to w_0, the parent will have his wealth decreased by w_0.

Thank you for your attention