A chaining algorithm for online nonparametric regression

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This is a joint work with Pierre Gaillard.

Introduction

- This talk is about regret minimization in a specific online learning problem known as online nonparametric regression.
- We consider the adversarial setting (almost no assumptions on the data sequence).
- We present an algorithm based on the chaining technique.

Outline of the talk:

1. The chaining technique in the stochastic setting
2. Our setting: online regression with individual sequences
3. Large (nonparametric) function sets
4. An algorithm based on the chaining technique

Bounding the expected supremum of a stochastic process

Technique introduced by Dudley (1967). Let $(X_f)_{f\in\mathcal{F}}$ be a centered stochastic process (indexed by a finite metric space $(\mathcal{F}, d)$) with subgaussian increments:

$$\forall f, g \in \mathcal{F}, \quad \forall \lambda > 0, \quad \log \mathbb{E} e^{\lambda(X_f - X_g)} \leqslant \frac{\lambda^2}{2} d(f,g)^2 \ .$$

Goal: upper bound the quantity $\mathbb{E}\big[\sup_{f\in\mathcal{F}} X_f\big] = \mathbb{E}\big[\sup_{f\in\mathcal{F}}(X_f - X_{f_0})\big]$ for any $f_0 \in \mathcal{F}$.

Lemma (see, e.g., Boucheron et al. 2013)

Let $Z_1, \ldots, Z_N$ be such that $\log \mathbb{E} e^{\lambda Z_i} \leqslant \lambda^2 v/2$ for all $\lambda > 0$ and $i \in [N]$. Then, $\mathbb{E} \max_{i=1,\ldots,N} Z_i \leqslant \sqrt{2v \log N}$.

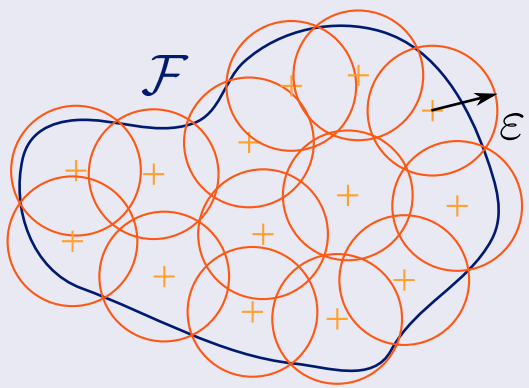
This lemma entails the pessimistic bound (correlations are not used): $\mathbb{E}\big[\sup_{f\in\mathcal{F}}(X_f - X_{f_0})\big] \leqslant B\sqrt{2\log(\operatorname{card}\mathcal{F})}$ with $B = \sup_{f\in\mathcal{F}} d(f, f_0)$.

Discretizing the space $(\mathcal{F}, d)$ into small balls

Definition (metric entropy)

- Let $(\mathcal{F}, d)$ be a metric space of finite cardinality.
- ε-net: any subset $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\forall f \in \mathcal{F},\ \exists g \in \mathcal{G} :\ d(f,g) \leqslant \varepsilon \quad \Longleftrightarrow \quad \bigcup_{g \in \mathcal{G}} \bar{B}(g, \varepsilon) = \mathcal{F}$$



- $\mathcal{N}_d(\mathcal{F}, \varepsilon)$: smallest cardinality of an ε-net.
- metric entropy of $\mathcal{F}$ at scale ε: $\log \mathcal{N}_d(\mathcal{F}, \varepsilon)$.
  It measures the complexity (richness) of the space $(\mathcal{F}, d)$.

Multi-scale discretization to exploit the correlations

Successive refining discretizations:
Let $\mathcal{F}^{(0)} = \{f_0\}, \mathcal{F}^{(1)}, \dots, \mathcal{F}^{(K-1)}, \mathcal{F}^{(K)} = \mathcal{F}$ be minimal $B/2^k$-nets of $\mathcal{F}$:

$$\forall f \in \mathcal{F},\ \exists \pi_k(f) \in \mathcal{F}^{(k)},\ d\big(f, \pi_k(f)\big) \leqslant B/2^k\ .$$

Chaining argument: using the lemma at multiple scales, we get:

$$\begin{aligned}
\mathbb{E}\Big[\sup_{f\in\mathcal{F}}(X_f - X_{f_0})\Big] &= \mathbb{E}\left[\sup_{f\in\mathcal{F}}\sum_{k=1}^{K}\Big(X_{\pi_k(f)} - X_{\pi_{k-1}(f)}\Big)\right] \\
&\leqslant \sum_{k=1}^{K}\mathbb{E}\left[\sup_{f\in\mathcal{F}}\Big(\underbrace{X_{\pi_k(f)} - X_{\pi_{k-1}(f)}}_{\text{small increments}}\Big)\right] \\
&\leqslant 6\sum_{k=1}^{K} B2^{-k}\sqrt{\log \mathcal{N}_d(\mathcal{F}, B/2^k)} \\
&\leqslant 12\int_0^{B/2}\sqrt{\log \mathcal{N}_d(\mathcal{F}, \varepsilon)}\, d\varepsilon\ .
\end{aligned}$$

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\end{aligned}$$

Setting: online regression with individual sequences

Prediction task: at each time $t \in \mathbb{N}^*$, predict the observation $y_t \in \mathbb{R}$ from the input $x_t \in \mathcal{X}$, on the basis of the past data $(x_1, y_1), \ldots, (x_{t-1}, y_{t-1})$.

Initial step: the environment chooses arbitrary deterministic sequences $(y_t)_{t \geqslant 1}$ in $\mathbb{R}$ and $(x_t)_{t \geqslant 1}$ in $\mathcal{X}$ but the forecaster has not access to them.

At each time round $t \in \mathbb{N}^*$,

1. The environment reveals the input $x_t \in \mathcal{X}$.
2. The forecaster chooses a prediction $\widehat{y}_t \in \mathbb{R}$.
3. The environment reveals the observation $y_t \in \mathbb{R}$ and the forecaster incurs the loss $(y_t - \widehat{y}_t)^2$.

Goal: minimizing regret

Let $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$ be a set of functions.

Goal of the forecaster: on the long run, to predict almost as well as the best function $f \in \mathcal{F}$ in hindsight, that is, to minimize the regret:

$$\mathrm{Reg}_T(\mathcal{F}) \triangleq \sum_{t=1}^{T} \big(y_t - \widehat{y}_t\big)^2 - \inf_{f \in \mathcal{F}} \sum_{t=1}^{T} \big(y_t - f(x_t)\big)^2 \ .$$

Individual sequence setting: our goal is to minimize the regret $\mathrm{Reg}_T(\mathcal{F})$ uniformly over all sequences $(y_t)_{t \geqslant 1}$ in $[-B, B]$ and $(x_t)_{t \geqslant 1}$ in $\mathcal{X}$; typically:

$$\sup_{\substack{|y_t| \leqslant B \\ x_t \in \mathcal{X}}} \left\{ \frac{1}{T} \sum_{t=1}^{T} (y_t - \widehat{y}_t)^2 - \inf_{f \in \mathcal{F}} \frac{1}{T} \sum_{t=1}^{T} (y_t - f(x_t))^2 \right\} \leqslant o(1) \quad \text{when } T \to +\infty \, .$$

Particular case: finite $\mathcal{F}$

Assume that $\mathcal{F} = \{f_1, f_2, \ldots, f_N\} \subseteq \mathbb{R}^{\mathcal{X}}$ is finite. We can use a well-known algorithm studied, e.g., by Kivinen and Warmuth (1999) and Vovk (2001):

Algorithm (Exponentially Weighted Average forecaster (EWA))

Parameter: $\eta > 0$

At each round $t \geqslant 1$,

- *Using past data, compute the weight vector* $\widehat{\boldsymbol{w}}_t = (\widehat{w}_{t,1}, \ldots, \widehat{w}_{t,N})$ *as*

$$\widehat{w}_{t,j} \triangleq \frac{\exp\left(-\eta \sum_{s=1}^{t-1} (y_s - f_j(x_s))^2\right)}{\sum_{j'=1}^{N} \exp\left(-\eta \sum_{s=1}^{t-1} (y_s - f_{j'}(x_s))^2\right)}, \quad 1 \leqslant j \leqslant N ;$$

- *Compute the convex combination (convex aggregate):*

$$\widehat{y}_t \triangleq \sum_{j=1}^{N} \widehat{w}_{t,j} \, f_j(x_t) .$$

Regret guarantee when $\mathcal{F}$ is finite

If $\mathcal{F}$ contains N functions, then we have a $\mathcal{O}(\log N)$ upper bound on the regret under the boundedness assumption:

$$|y_1|, \ldots, |y_T| \leqslant B \qquad \text{and} \qquad \|f_1\|_\infty, \ldots, \|f_N\|_\infty \leqslant B \ .$$

Theorem (Kivinen and Warmuth 1999)

Assume that $\mathcal{F} = \{f_1, f_2, \ldots, f_N\} \subseteq [-B, B]^{\mathcal{X}}$.

Then, the EWA algorithm tuned with $\eta = 1/(8B^2)$ *satisfies: for all sequences* $(y_t)_{t \geqslant 1}$ *in* $[-B, B]$ *and* $(x_t)_{t \geqslant 1}$ *in* $\mathcal{X}$, *for all* $T \geqslant 1$,

$$\sum_{t=1}^{T} \big(y_t - \widehat{y}_t\big)^2 - \min_{1 \leqslant j \leqslant N} \sum_{t=1}^{T} \big(y_t - f_j(x_t)\big)^2 \leqslant 8B^2 \log N \ .$$

Remark 1: the requirement $\forall j, \|f_j\|_\infty \leqslant B$ can be removed via clipping.

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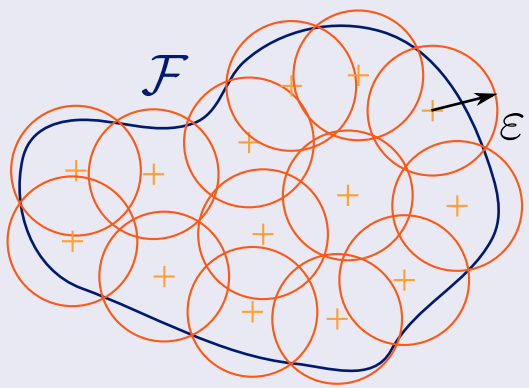
Remark 1: the requirement $\forall j, \|f_j\|_\infty \leqslant B$ can be removed via clipping.
Remark 2: we can obtain a similar bound if $B = \max_{1 \leqslant t \leqslant T} |y_t|$ is unknown.

Large function sets $\mathcal{F}$: finite approximation

Definition (metric entropy for sup norm)

- Let $\mathcal{F} \subseteq \mathbb{R}^{\mathcal{X}}$ be a set of bounded functions endowed with the sup norm $\|f\|_\infty \triangleq \sup_{x \in \mathcal{X}} |f(x)|$.
- ε-net: any subset $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\forall f \in \mathcal{F},\ \exists g \in \mathcal{G} :\ \|f - g\|_\infty \leqslant \varepsilon\ .$$



- $\mathcal{N}_\infty(\mathcal{F}, \varepsilon)$: smallest cardinality of an ε-net.
- metric entropy of $\mathcal{F}$ at scale ε: $\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon)$.

Large function sets $\mathcal{F}$: finite approximation (2)

Assume that $\mathcal{F}$ is infinite (the EWA algorithm cannot be used). Small regret is still achievable if $\mathcal{F}$ can be well approximated by a finite set.

Discretizing $\mathcal{F}$ (Vovk, 2006): approximate $\mathcal{F}$ with a minimal ε-net and run the EWA algorithm on this finite subset:

$$\sum_{t=1}^{T}\big(y_t-\widehat{y}_t\big)^2 \leqslant \min_{1\leqslant j\leqslant \mathcal{N}_\infty(\mathcal{F},\varepsilon)} \sum_{t=1}^{T}\big(y_t-f_j(x_t)\big)^2 + 8B^2\log\mathcal{N}_\infty(\mathcal{F},\varepsilon)$$

$$\leqslant \inf_{f\in\mathcal{F}} \sum_{t=1}^{T}\big(y_t-f(x_t)\big)^2 + T\varepsilon^2 + 4TB\varepsilon + 8B^2\log\mathcal{N}_\infty(\mathcal{F},\varepsilon)$$

Finite-dimensional case: given $\varphi_j : \mathcal{X}\to[-B,B]$ and a compact set $\Theta\subseteq\mathbb{R}^d$, define

$$\mathcal{F}=\left\{\sum_{j=1}^{d}\theta_j\varphi_j : \theta\in\Theta\right\}\subseteq\mathbb{R}^{\mathcal{X}}\ .$$

Note that $\mathcal{N}_\infty(\mathcal{F},\varepsilon)\lesssim(1/\varepsilon)^d$. Choosing $\varepsilon\approx 1/T$ yields a regret at most of the order of $d\log(T)$, which is optimal (parametric rate).

What if $\mathcal{F}$ is very large (nonparametric)?

Nonparametric set: assume that $\mathcal{F}$ is much larger than in the finite-dimensional case:

$$\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon) \approx (1/\varepsilon)^p \qquad \text{as} \qquad \varepsilon \to 0 \ .$$

Example: Hölder class $\mathcal{F} \subseteq \mathbb{R}^{[0,1]}$ of regularity $\beta = q + \alpha$:

$$\left|f^{(q)}(x) - f^{(q)}(y)\right| \leqslant \lambda |x-y|^\alpha \quad \text{and} \quad \forall k \leqslant q, \ \|f^{(k)}\|_\infty \leqslant B$$

In this case, $\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon) \approx \varepsilon^{-1/\beta}$ so that $p = 1/\beta$.

EWA is suboptimal: the regret bound $T\varepsilon^2 + 4TB\varepsilon + 8B^2 \log \mathcal{N}_\infty(\mathcal{F}, \varepsilon)$ becomes roughly $T\varepsilon + (1/\varepsilon)^p$. Optimizing in ε only yields:

$$\sum_{t=1}^T \big(y_t - \widehat{y}_t\big)^2 \leqslant \inf_{f \in \mathcal{F}} \sum_{t=1}^T \big(y_t - f(x_t)\big)^2 + \mathcal{O}\big(T^{p/(p+1)}\big) \ ,$$

which is worse than the optimal rate $\mathcal{O}\big(T^{p/(p+2)}\big)$.

Optimal rates by Rakhlin and Sridharan (2014)

We still assume that $\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon) \approx (1/\varepsilon)^p$ as $\varepsilon \to 0$.

Optimal regret: through a non-constructive approach (reduction to a stochastic problem via von Neumann minimax theorem), Rakhlin and Sridharan (2014) proved that, if $p \in (0, 2)$, then

$$\begin{aligned}
\mathrm{Reg}_T(\mathcal{F}) &\leqslant c_1 B^2 \big(1 + \log \mathcal{N}_\infty(\mathcal{F}, \gamma)\big) + c_2 B \sqrt{T} \int_0^\gamma \sqrt{\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon)} d\varepsilon \\
&\lesssim \gamma^{-p} + \sqrt{T} \int_0^\gamma \varepsilon^{-p/2} d\varepsilon \\
&\lesssim T^{p/(p+2)} \qquad \text{for } \gamma = T^{-1/(p+2)}.
\end{aligned}$$

The rate $T^{p/(p+2)}$ is better than $T^{p/(p+1)}$ obtained previously with EWA, and it is (in a sense) optimal.

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&\lesssim T^{p/(p+2)} \qquad \text{for } \gamma = T^{-1/(p+2)}.
\end{aligned}$$

Example (Hölder class with regularity β):
Since $p = 1/\beta$, we get $\mathrm{Reg}_T(\mathcal{F})/T = \mathcal{O}\big(T^{-2\beta/(2\beta+1)}\big)$ if $\beta > 1/2$.
Therefore, same rate as in the statistical setting (for $\beta > 1/2$).

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&\lesssim T^{p/(p+2)} \qquad \text{for } \gamma = T^{-1/(p+2)}.
\end{aligned}$$

The above integral is a Dudley entropy integral.

- In statistical learning with i.i.d. data, useful to derive risk bounds for empirical risk minimizers (e.g., Massart 2007; Rakhlin et al. 2013).
- Also appears in online learning with individual sequences. Earlier appearances: Opper and Haussler (1997); Cesa-Bianchi and Lugosi (1999, 2001).

Our contributions

1. We provide an explicit algorithm that achieves the Dudley-type regret bound (when $p \in (0,2)$):

$$\mathrm{Reg}_T(\mathcal{F}) \leqslant c_1 B^2 \big(1+\log \mathcal{N}_\infty(\mathcal{F},\gamma)\big)+c_2 B\sqrt{T}\int_0^\gamma \sqrt{\log \mathcal{N}_\infty(\mathcal{F},\varepsilon)}d\varepsilon\,.$$

   Nota: contrary to Rakhlin and Sridharan (2014), our bounds are not in terms of the stronger notion of *sequential entropy*.

2. This algorithm uses ideas from the chaining technique, and relies on a new subroutine (Multi-variable Exponentiated Gradient algorithm) to perform optimization at different scales simultaneously.

3. We address computational issues by showing how to construct more efficient and quasi-optimal ε-nets (for Hölder classes).

Linearizing the square loss can help locally (1)

Suppose we play with loss functions $\boldsymbol{u} \mapsto \ell_t(\boldsymbol{u})$, $t \geqslant 1$, that are convex and differentiable over the simplex $\Delta_N = \left\{\boldsymbol{u} \in \mathbb{R}_+^N : \sum_{i=1}^N u_i = 1\right\}$.

Algorithm (Exponentiated Gradient—EG)

Parameter: $\eta > 0$

At each round $t \geqslant 1$, *compute the weight vector* $\widehat{\boldsymbol{u}}_t \in \Delta_N$ *by*

$$\widehat{u}_{t,j} \triangleq \frac{1}{Z_t} \exp\left(-\eta \sum_{s=1}^{t-1} \partial_{\widehat{u}_{s,j}} \ell_s(\widehat{\boldsymbol{u}}_s)\right) \quad , \quad 1 \leqslant j \leqslant N \ .$$

Theorem (Kivinen and Warmuth 1999 and Cesa-Bianchi 1999)

Assume ℓ_t *convex, diff, and* $\|\nabla \ell_t\|_\infty \leqslant G$. *For* $\eta = G^{-1}\sqrt{2\log(N)/T}$,

$$\sum_{t=1}^T \ell_t\big(\widehat{\boldsymbol{u}}_t\big) \leqslant \min_{\boldsymbol{u} \in \Delta_N} \sum_{t=1}^T \ell_t(\boldsymbol{u}) + G\sqrt{2T\log N} \ .$$

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Linearizing the square loss can help locally (2)

Application: we want to predict almost as well as the best function in $\mathcal{F} = \{f_0 + g_j : j = 1, \dots, N\}$ with $\|g_j\|_\infty$ small (neighbors of f_0).

We use EG with $\ell_t(\boldsymbol{u}) = \left(y_t - f_0(x_t) - \sum_{j=1}^N u_j g_j(x_t)\right)^2$, $\boldsymbol{u} \in \Delta_N$.

Since $\|\nabla \ell_t\|_\infty \lesssim B \max_j \|g_j\|_\infty$, the EG algorithm satisfies:

$$\sum_{t=1}^T \Bigg(y_t - \underbrace{f_0(x_t) - \sum_{j=1}^N \widehat{u}_{t,j} g_j(x_t)}_{=\widehat{y}_t} \Bigg)^2 \leqslant \min_{1 \leqslant j \leqslant N} \sum_{t=1}^T (y_t - f_0(x_t) - g_j(x_t))^2 + \square B \max_{1 \leqslant j \leqslant N} \|g_j\|_\infty \sqrt{T \log N}$$

Advantage: the above regret bound $B \max_j \|g_j\|_\infty \sqrt{T \log N}$ improves on $B^2 \log N$ (obtained by EWA) when $\max_j \|g_j\|_\infty \ll B\sqrt{\log(N)/T}$.

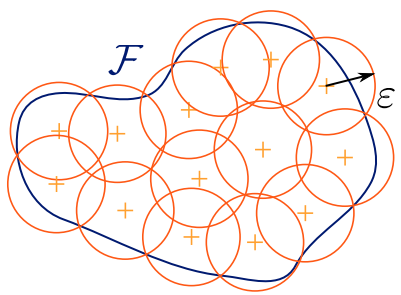
Thus, linearizing the square loss can help if the functions in $\mathcal{F}$ are close.

Turning the chaining technique into an online algorithm

We still assume that $\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon) \approx (1/\varepsilon)^p$ as $\varepsilon \to 0$. Recall that we want to prove a bound of the form:

$$\sum_{t=1}^{T} \big(y_t - \widehat{y}_t\big)^2 \leqslant \inf_{f \in \mathcal{F}} \sum_{t=1}^{T} \big(y_t - f(x_t)\big)^2 + [\text{small term}]$$

Chaining principle: as previously, we discretize $\mathcal{F}$ and use projections $\pi_k(f)$ such that $\sup_f \|\pi_k(f) - f\|_\infty \leqslant \gamma/2^k$ for all $k \geqslant 0$.



$$\inf_{f \in \mathcal{F}} \sum_{t=1}^{T} \big(y_t - f(x_t)\big)^2 = \inf_{f \in \mathcal{F}} \sum_{t=1}^{T} \left(y_t - \pi_0(f)(x_t) - \sum_{k=1}^{\infty} \underbrace{\big[\pi_k(f) - \pi_{k-1}(f)\big](x_t)}_{|\text{small increments}| \leqslant 3\gamma/2^k} \right)^2$$

Aggregation at two different levels

$$\inf_{f\in\mathcal{F}}\sum_{t=1}^{T}\big(y_t-f(x_t)\big)^2=\inf_{f\in\mathcal{F}}\sum_{t=1}^{T}\Bigg(y_t-\underbrace{\pi_0(f)}_{\in\mathcal{F}^{(0)}}(x_t)-\sum_{k=1}^{\infty}\underbrace{\big[\pi_k(f)-\pi_{k-1}(f)\big]}_{\in\mathcal{G}^{(k)}}(x_t)\Bigg)^2$$

Sufficient goal:

$$\sum_{t=1}^{T}\big(y_t-\widehat{y}_t\big)^2\leqslant\inf_{f_0,g_1,\ldots,g_K}\sum_{t=1}^{T}\big(y_t-(f_0+g_1+\ldots+g_K)(x_t)\big)^2+[\text{small term}]$$

Two aggregation levels:

$$\left.\begin{array}{lcl} f_{0,1} & \xrightarrow{\substack{\text{low scale}\\ \text{multi-variable EG}}} & \widehat{f}_{t,1}\\ f_{0,2} & \longrightarrow & \widehat{f}_{t,2}\\ \vdots & & \vdots\\ f_{0,N_0} & \longrightarrow & \widehat{f}_{t,N_0}\end{array}\right\}\xrightarrow{\substack{\text{high scale}\\ \text{EWA}}}\widehat{y}_t=\sum_{j=1}^{N_0}\widehat{w}_{t,j}\widehat{f}_{t,j}(x_t)$$

Combining two regret guarantees

High-scale aggregation Using an Exponentially Weighted Average (EWA) forecaster $\widehat{f}_t = \sum_{j=1}^{N_0} \widehat{w}_{t,j} \widehat{f}_{t,j}$ yields

$$\sum_{t=1}^{T} (y_t - \widehat{y}_t)^2 \leqslant \min_{1 \leqslant j \leqslant N_0} \sum_{t=1}^{T} \left(y_t - \widehat{f}_{t,j}(x_t) \right)^2 + \square B^2 \log N_0$$

Low-scale aggregation Recall that $\mathcal{G}^{(k)} = \{\pi_k(f) - \pi_{k-1}(f) : \ f \in \mathcal{F}\}$. Denote $\mathcal{G}^{(k)} = \left\{ g_1^{(k)}, \ldots, g_{N_k}^{(k)} \right\}$.

We designed a multi-variable extension of the Exponentiated Gradient algorithm:

$$\widehat{f}_{t,j} \triangleq f_{0,j} + \sum_{k=1}^{K} \sum_{i=1}^{N_k} \widehat{u}_{t,i}^{(j,k)} g_i^{(k)}$$

which yields, for all $j = 1, \ldots, N_0$,

$$\sum_{t=1}^{T} \left(y_t - \widehat{f}_{t,j}(x_t) \right)^2 \leqslant \min_{g_1, \ldots, g_K} \sum_{t=1}^{T} \left(y_t - \left(f_{0,j} + g_1 + \ldots + g_K \right)(x_t) \right)^2 + 120 B \sqrt{T} \int_0^{\gamma/2} \sqrt{\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon)} d\varepsilon \ .$$

Main result

The next theorem indicates that the Chaining Exponentially Weighted Average forecaster satisfies a Dudley-type regret bound.

Theorem (Gaillard and G., 2015)

Let $B > 0$, $T \geqslant 1$, *and* $\gamma \in \left(\frac{B}{T}, B\right)$.

- *Assume that* $\max_{1 \leqslant t \leqslant T} |y_t| \leqslant B$ *and that* $\sup_{f \in \mathcal{F}} \|f\|_\infty \leqslant B$.
- *Assume that* $(\mathcal{F}, \|\cdot\|_\infty)$ *is totally bounded and define* $\mathcal{F}^{(0)}$ *and* $\mathcal{G}^{(k)}$ *as above.*

Then, the Chaining Exponentially Weighted Average forecaster (tuned with appropriate parameters) satisfies:

$$\mathrm{Reg}_T(\mathcal{F}) \leqslant B^2\left(5 + 50 \log \mathcal{N}_\infty(\mathcal{F}, \gamma)\right) + 120 B \sqrt{T} \int_0^{\gamma/2} \sqrt{\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon)}\, d\varepsilon \,.$$

Computational issues: dyadic discretization

We assume that $\mathcal{F} = \{f : [0,1] \to [-B, B] : \ f \text{ is 1-Lipschitz}\}$.

Regret bound:
We know that $\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon) = \mathcal{O}(\varepsilon^{-1})$.
Therefore, our algorithm obtains $\mathrm{Reg}_T(\mathcal{F}) = \mathcal{O}(T^{1/3})$, which is optimal.

Computational issue:
Our algorithm updates $\exp(\mathcal{O}(T))$ weights at every round t.
Hence very poor time and space computational complexities.

Solution:
$\mathcal{F}$ has a sufficiently nice structure that can be exploited to construct computationally manageable ε-nets with quasi-optimal cardinality.

For example: piecewise-constant approximations on a dyadic discretization lead to $\mathcal{O}(T^{1/3} \log T)$ regret and per-round time complexity.

Conclusion

- We designed an explicit algorithm with a Dudley-type regret bound for online nonparametric regression.
- We provided an efficient implementation for Hölder classes.

Thank you for your attention!

Appendix

5 Computational issues: dyadic discretization

Lipschitz class $\mathcal{F}$: a computationally efficient discretization

We assume that $\mathcal{F} = \{f : [0,1] \to [-B, B] : \ f \text{ is 1-Lipschitz}\}$.

Regret bound:
We know that $\log \mathcal{N}_\infty(\mathcal{F}, \varepsilon) = \mathcal{O}(\varepsilon^{-1})$.
Therefore, our algorithm obtains $\mathrm{Reg}_T(\mathcal{F}) = \mathcal{O}(T^{1/3})$, which is optimal.

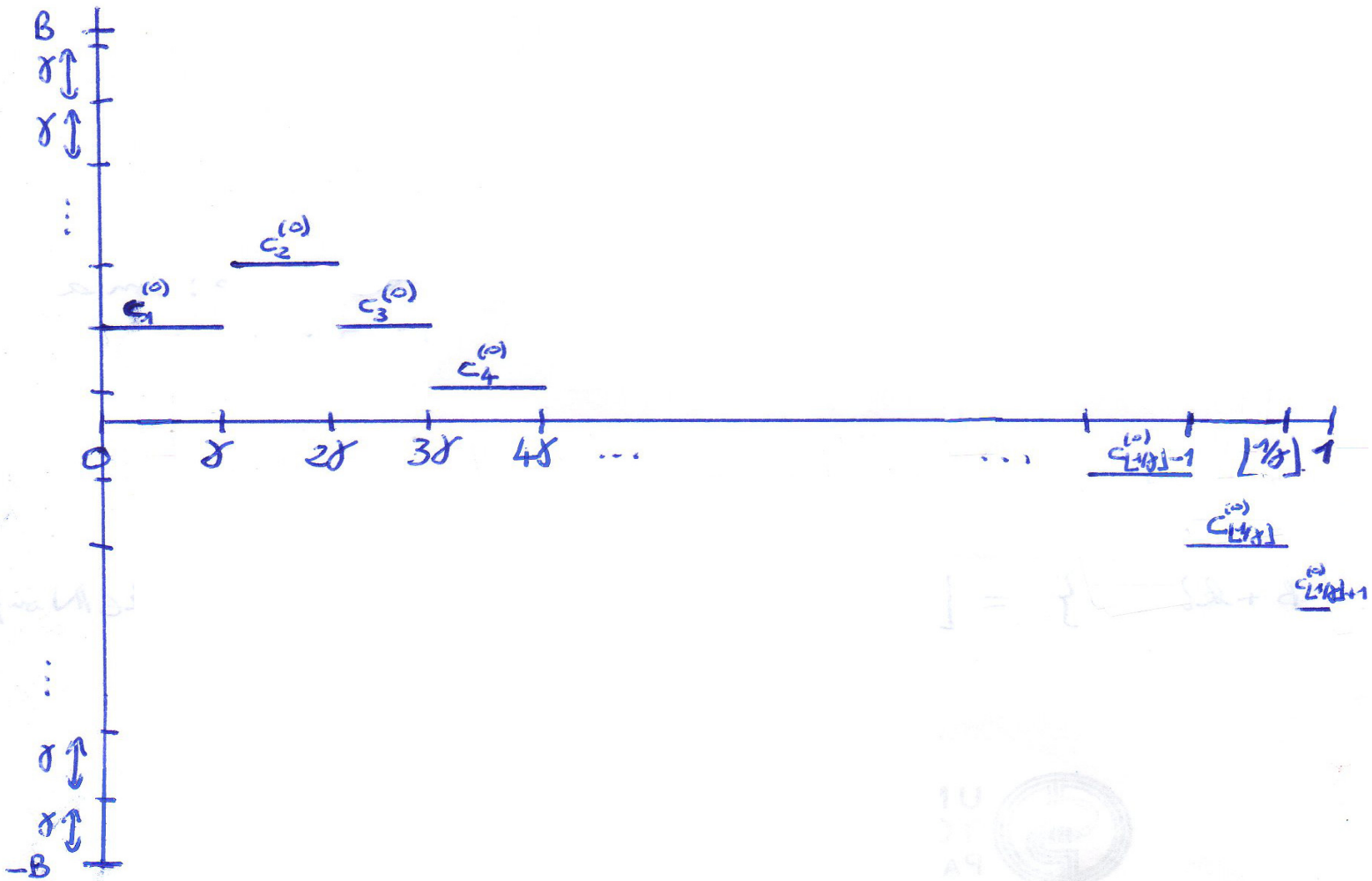
Computational issue:
Our algorithm updates exponentially many weights at every round t.
Hence poor time and space computational complexities.

Solution:
$\mathcal{F}$ has a sufficiently nice structure that can be exploited to construct computationally manageable ε-nets with quasi-optimal cardinality.

High-level discretization (piecewise-constant approximation)

- Partition the x-axis $[0,1]$: $I_a \triangleq \big[(a-1)\gamma, a\gamma\big)$, $a = 1, \ldots, \frac{1}{\gamma}$.
- Discretize the y-axis $[-B, B]$: $\mathcal{C}^{(0)} = \big\{-B + j\gamma : \; j = 0, \ldots, \frac{2B}{\gamma}\big\}$.

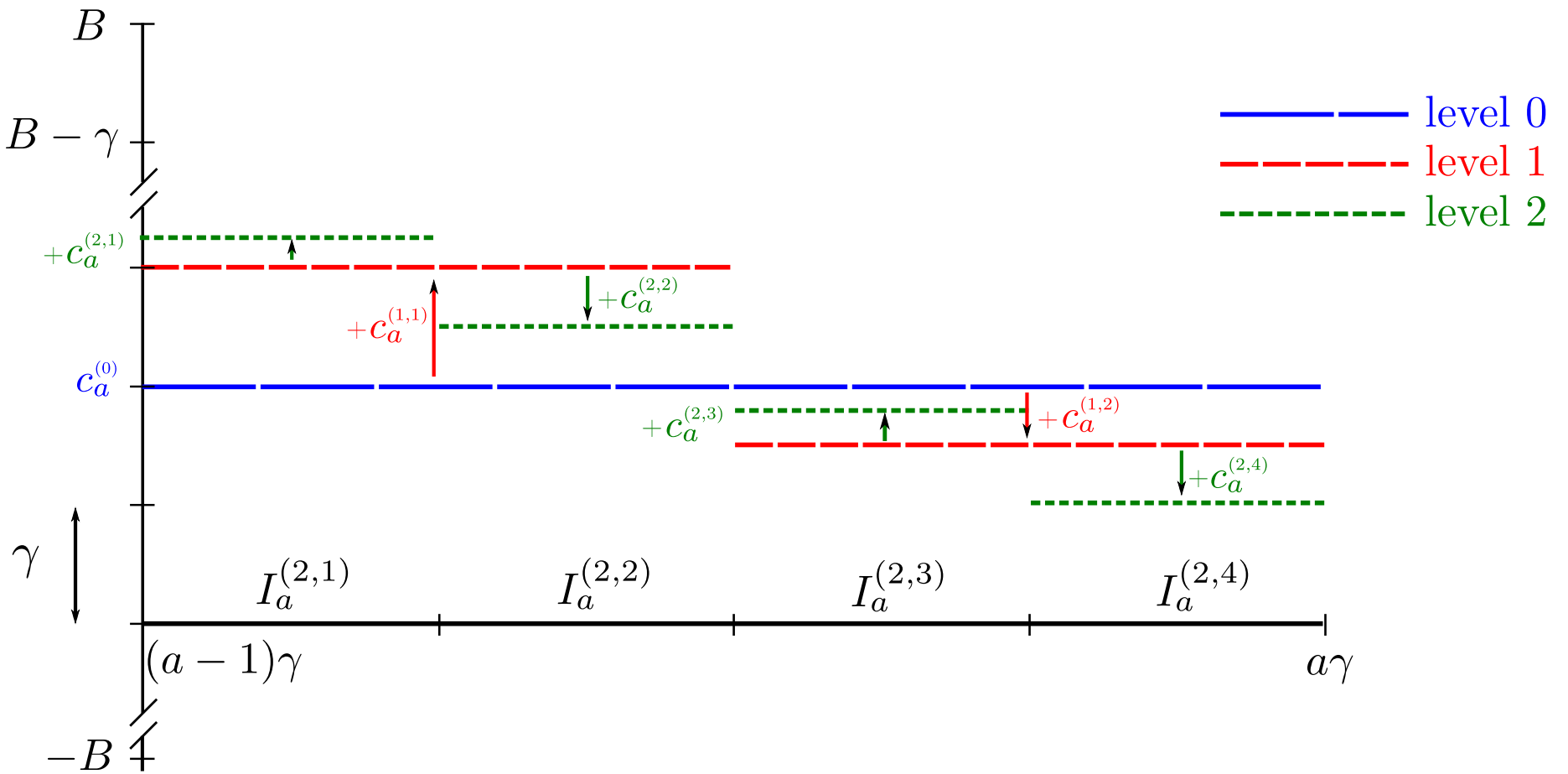
$\mathcal{F}^{(0)}$: set of piecewise-constant functions $f^{(0)}(x) = \sum_{a=1}^{1/\gamma} c_a^{(0)} \mathbb{I}_{x \in I_a}$, $c_a^{(0)} \in \mathcal{C}^{(0)}$.



Low-level discretization (dyadic approximation)

$\mathcal{F}^{(M)}$: set of all functions $f_c : [0,1] \to \mathbb{R}$ of the form

$$f_c(x) = \underbrace{\sum_{a=1}^{1/\gamma} c_a^{(0)} \mathbb{I}_{x \in I_a}}_{f^{(0)}(x)} + \underbrace{\sum_{m=1}^{M} \sum_{a=1}^{1/\gamma} \sum_{n=1}^{2^m} c_a^{(m,n)} \mathbb{I}_{x \in I_a^{(m,n)}}}_{g^{(m)}(x)} .$$



Regret and computational efficiency

Theorem (Gaillard and G., 2015)

Let $B > 0$, $T \geqslant 2$, and $\mathcal{F}$ be the set of all 1-Lipschitz functions from $[0,1]$ to $[-B,B]$. Assume that $\max_{1 \leqslant t \leqslant T} |y_t| \leqslant B$.

Then, the Dyadic Chaining Algorithm (see preprint) satisfies, for some absolute constant $c > 0$,

$$\mathrm{Reg}_T(\mathcal{F}) \leqslant c \max\{B, B^2\} T^{1/3} \log T \ .$$

Remark: additional log factor, but computationally tractable:

- per-round time complexity: $\mathcal{O}\big(T^{1/3} \log T\big)$;
- space complexity: $\mathcal{O}\big(T^{4/3} \log T\big)$.

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