Market games

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Financial market $\rightarrow$ Repeated game with incomplete information.

- Traders have different levels of information.
- Traders act strategically, use powerful automatons
- High frequency trading: traders can give orders every 0.1 ms

Market	1 asset	Several assets
Risk Neutral	De Meyer - M. Saley [2002] De Meyer [2010]	Gensbittel [2009]
Risk Averse	De Meyer - Fournier [2015]	Fournier [2015]

De Meyer - Fournier 2015

Player 1

- Better informed
- Risk neutral

Player 2

- Less informed
- Risk averse

De Meyer - Fournier 2015

Saint-Petersbourg game

You pay an entry fee x€ and you flip a fair coin:

tail you leave the game with 2^0 €
head, you flip the coin again:

tail you leave the game with 2^1 €
head, you flip the coin again:

tail you leave the game with 2^2 €
head, you flip the coin again, etc

...

With probability $\frac{1}{2^{n+1}}$ you leave the game with 2^n €.

Player 2

- Less informed
- Risk averse

De Meyer - Fournier 2015

Risk neutral: $x = \sum_{n \geq 0} \frac{1}{2^{n+1}} 2^n = \infty$

Risk averse: if m euros gives you an utility of $log(n)$:

$\sum_{n \leq 0} \frac{log(2^n)}{2^{n+1}} = log(2)$

Player 2

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- At the beginning of the game, the final value L of a risky asset is drawn according to a distribution law μ. Player 1 knows L. Player 2 only knows the law μ.

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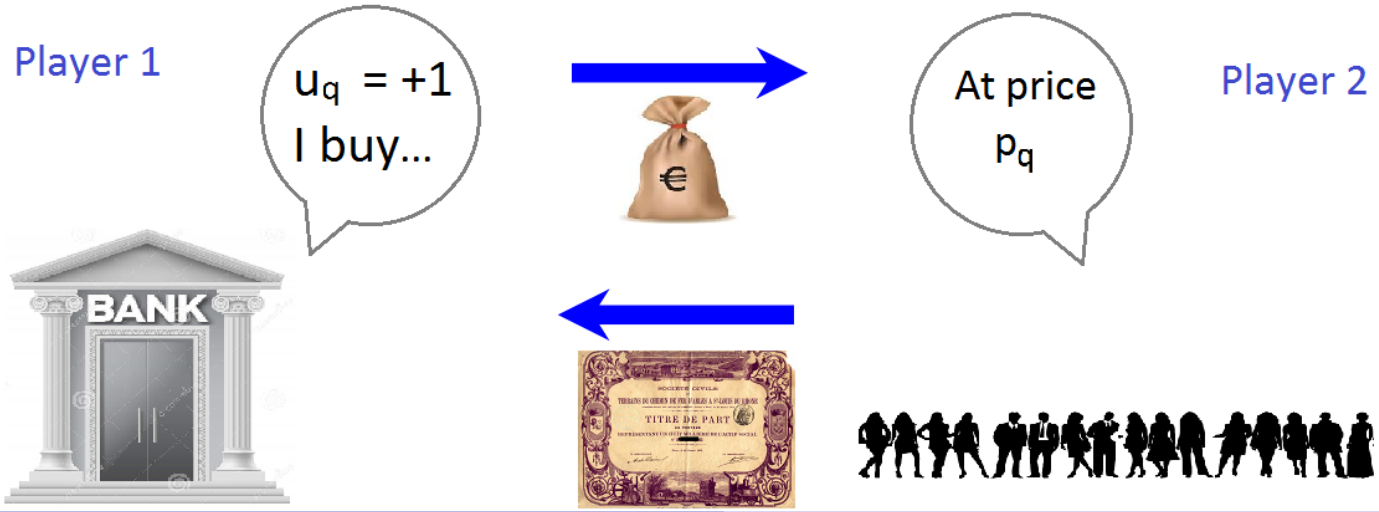
- At each period $q \in \{1, \dots, n\}$, players know the past actions and exchange one unit of the risky asset against money: player 1 decides whether he wants to buy ($u_q = 1$) or to sell ($u_q = -1$) one unit of the risky asset at this period. Simultaneously, player 2 selects the price $p_q \in \mathbb{R}$ of the transaction at stage q.



Behavioral strategies

At period $q \in \{1, \dots, n\}$, players have observed past actions $h_{q-1} := (u_1, p_1, \dots, u_{q-1}, p_{q-1})$.

- Behavioral strategy of player 1: $\sigma = (\sigma_1, \dots, \sigma_n)$ where $\sigma_q : (h_{q-1}, L) \to \Delta(\{-1, +1\})$.

- Behavioral strategy of player 2: $\tau = (\tau_1, \dots, \tau_n)$ where $\tau_q : h_{q-1} \to \Delta(\mathbb{R})$.

We say that a strategy is **reduced** if it doesn't depend on the past actions of player 2.

The payoff

After n stages, the portfolio of player 1 is:

$$\begin{array}{c} \sum_{q=1}^{n} u_q \text{ shares of risky asset} \\ \sum_{q=1}^{n} -u_q p_q \text{ quantity of money.} \end{array}$$

$$\begin{cases} g_1(\sigma,\tau) = E_{\mu,\sigma,\tau}\left[\sum_{q=1}^{n} u_q L - u_q p_q\right] \\ g_2(\sigma,\tau) = E_{\mu,\sigma,\tau}\left[\hat{H}\left(\sum_{q=1}^{n} u_q p_q - u_q L\right)\right] \end{cases}$$

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$$\begin{cases} g_1(\sigma,\tau) = E_{\mu,\sigma,\tau}\left[\frac{1}{\sqrt{n}}\sum_{q=1}^{n} u_q L - u_q p_q\right] \\ g_2(\sigma,\tau) = E_{\mu,\sigma,\tau}\left[H\left(\frac{1}{\sqrt{n}}\sum_{q=1}^{n} u_q L - u_q p_q\right)\right] \end{cases}$$

Smoothness hypothesis

- μ is a probability measure on a compact interval, absolutely continuous with respect to the Lebesgue measure. Its density function f_μ is strictly positive and C^1.
- H is strictly positive, strictly convex and C^2. H' is Lipschitz and bounded.

The main results

Under smoothness hypothesis on μ and H, we have:

1. Existence of a Nash equilibrium in the n-repeated game.
   - $\rightarrow$ Players play reduced strategies.
   - $\rightarrow$ Player 1 plays a completely mixed strategy.
   - $\rightarrow$ Player 2 plays a pure strategy.

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1. Existence of a Nash equilibrium in the n-repeated game.
   - $\rightarrow$ Players play reduced strategies.
   - $\rightarrow$ Player 1 plays a completely mixed strategy.
   - $\rightarrow$ Player 2 plays a pure strategy.
2. Any equilibrium strategy of player 2 $(p_1, p_2(u_1) \ldots, p_n(u_1, \ldots, u_{n-1}))$ has another property: suppose that $(u_1, \ldots, u_n) \sim \lambda_n$ (the uniform distribution), then $(p_1, p_2(u_1) \ldots, p_n(u_1, \ldots, u_{n-1}))$ is a martingale.

Martingales and Arbitrage

Harisson Pliska [1981]: Evaluation of assets under the theory of arbitrage-free.

➪ In an arbitrage free market ($\simeq$ where it is impossible to win money without taking any risk), there exists an equivalent martingale probability.

➪ This martingale is a powerful tool, widely used in mathematical finance.

The main results

3. Suppose that $(u_1, \ldots, u_n)$ are uniformly distributed, and that players play more and more frequently:

   $\rightarrow$ the limit price process in continuous time is also a martingale.
   $\rightarrow$ this limit martingale is the same for every equilibrium
   $\rightarrow$ this limit martingale is continuous and can be written $f(B_t, t)$ where B is a standard Brownian motion and $f : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ is a deterministic function, increasing with respect to its first variable.

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➥ This class of dynamics is compatible with the models currently used in financial markets (Black-Scholes [1973]...).

Continuous Martingale of Maximal Variation

The n-variation of a martingale $(X_t)_{t\in[0,1]}$ is

$$V_X^n = \sum_{q=0}^{n-1} \|X_{\frac{q+1}{n}} - X_{\frac{q}{n}}\|_{L^1}$$

Consider the problem $\mathcal{M}_n$ of maximizing the n-variation V_X^n on the class of martingales X with final distribution μ ($X_1 \sim \mu$).

The martingales that solve $\mathcal{M}_n$, (i.e. martingales of maximal n-variation) converge in finite distributions, as n goes to infinity, to a continuous martingale of maximal variation.

Sketch of the existence proof

1. Transform the strategy spaces.

   ▶ Player 1 can select at the beginning of the game a probability Π_n on the vector $(L, u_1, \ldots, u_n)$ with the marginal constraint $L \sim \mu$.

   ▶ Player 2 selects a convex function $\Psi_n : \mathbb{R} \to \mathbb{R}$, related to the price by the equation:

$$p_q(u_1, \ldots, u_{q-1}) = \sqrt{n} E_{\lambda_n} \left[u_q \Psi_n \left(\frac{\sum_{i=1}^n u_i}{\sqrt{n}} \right) | u_1, \ldots, u_{q-1} \right]$$

2. We show that a couple of strategies (Π_n, Ψ_n) is a Nash equilibrium if and only if the 4 following conditions are satisfied:

$$\begin{cases} \textbf{(C1)} \quad \text{By definition: } \Pi_{n|L} = \mu \\ \textbf{(C2)} \quad \text{Normalisation: } \Psi_n \text{ is such that } E_{\lambda_n}[\Psi_n(\frac{\sum_{i=1}^n u_i}{\sqrt{n}})] = 0 \\ \textbf{(C3)} \quad \text{FOC for player 1: } \Pi_n(L \in \partial\Psi_n(\frac{\sum_{i=1}^n u_i}{\sqrt{n}})) = 1 \\ \textbf{(C4)} \quad \text{FOC for player 2:} \\ \frac{\partial \Pi_{n|u_1,\dots,u_n}}{\partial \lambda_n} = \frac{\alpha_n}{E_{\Pi_n}[H'(LS_n - \Psi_n(\frac{\sum_{i=1}^n u_i}{\sqrt{n}}))|u_1,\dots,u_n]} \text{ where } \alpha_n > 0 \end{cases}$$

3. A partial solution: a profile of strategies that satisfies **(C1)**, **(C2)** et **(C3)**.

Lemma

*There exists a family of partial solutions: for every probability $\nu \in \Delta(\mathbb{R})$, there exists a couple (Π_ν, Ψ_ν) solution of **(C1)**, **(C2)** and **(C3)**.*

4. Can we find ν such that (Π_ν, Ψ_ν) also satisfies **(C4)**?

- Reformulation with an operator $T : \Delta_f(\mathbb{R}) \to \Delta_f(\mathbb{R})$.

$$\text{Can we find } \nu \text{ such that } T(\nu) = \rho \quad ?$$

- The operator T:

1/ Is continuous (for the second order Wasserstein distance)
2/ Satisfies the hypothesis $T(\nu) \ll \nu$

It is therefore onto, from $\Delta_f(\mathbb{R})$ to itself.

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It is therefore onto, from $\Delta_f(\mathbb{R})$ to itself.

➥ There exists a Nash equilibrium in $G_n(\mu)$.

The price martingale.

Suppose that $(u_1, \ldots, u_n) \sim \lambda_n$.

$$p_q(u_1, \ldots, u_{q-1}) = \sqrt{n} E_{\lambda_n} \left[u_q \Psi_{\nu_n} \left(\frac{\sum_{i=1}^n u_i}{\sqrt{n}} \right) | u_1, \ldots, u_{q-1} \right]$$

The price process is a martingale.

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➥ Under λ_n, the price process $(p_1, p_2, \dots)$ is a martingale.

Asymptotic results

1. The price at date q of the game with length n:

$$\begin{aligned} p_q^n =& \sqrt{n} E_{\lambda_n}\left[u_n \Psi_{\nu_n}\left(\frac{\sum_{i=1}^n u_i}{\sqrt{n}}\right) | u_1, \ldots, u_{q-1}\right] \\ =& \sqrt{n} E_{\lambda_n}\left[u_n \Psi_{\nu_n}\left(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} + \frac{u_n}{\sqrt{n}}\right) | u_1, \ldots, u_{q-1}\right] \end{aligned}$$

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$$= \frac{\sqrt{n}}{2} E_{\lambda_n}[\Psi_{\nu_n}(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}) - \Psi_{\nu_n}(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} - \frac{1}{\sqrt{n}}) \mid u_1, \dots, u_{q-1}]$$

2. Intuition of the convergence.

$$\frac{\sqrt{n}}{2} E_{\lambda_n}\left[\Psi_{\nu_n}\left(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}\right) - \Psi_{\nu_n}\left(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} - \frac{1}{\sqrt{n}}\right) \mid u_1, \ldots, u_{q-1}\right]$$

- Ψ_{ν_n} converges uniformly to Ψ_{ν_∞}
- For a fixed differentiable function Ψ, $\frac{\Psi(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}) - \Psi(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}} - \frac{1}{\sqrt{n}})}{2/\sqrt{n}}$ converges to $\Psi'(\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}})$.
- $\frac{\sum_{i=1}^{n-1} u_i}{\sqrt{n}}$ converges in law to B_1, where B is a standard Brownian motion.

Asymptotic results

3. The sequence $(\nu_n)_{n\geq 1}$ converges.

   - Compactness argument: every subsequence of ν_n has an accumulation point, which the solution of an equation.
   - Equivalence with a non linear differential system.
   - Uniqueness of solution to this system.

$p^n_{\lfloor nt \rfloor}$ is the price at fraction $t \in [0,1]$ of the game with length n.

Let $p_t := \lim_{n\to\infty} p^n_{\lfloor nt \rfloor}$, then: $p_t := E[\Psi_{\nu_\infty}{}' (B_1) \mid B_t]$

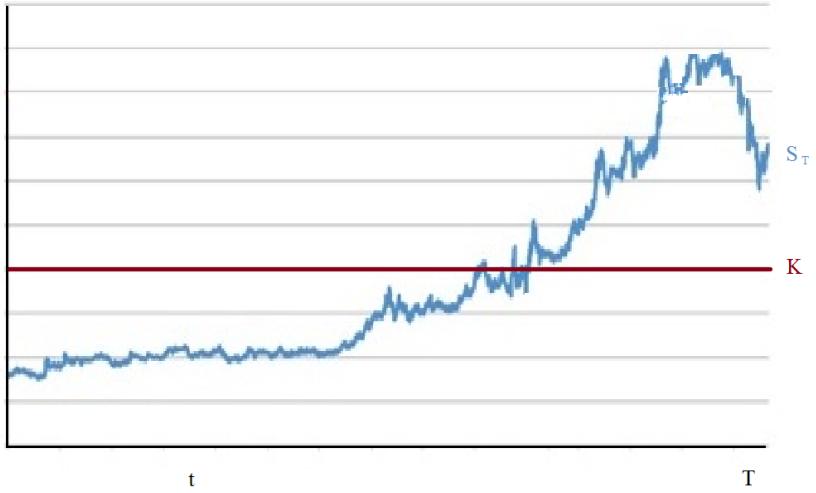
➥ Under the probability λ_n, the limit price process is a CMMV.

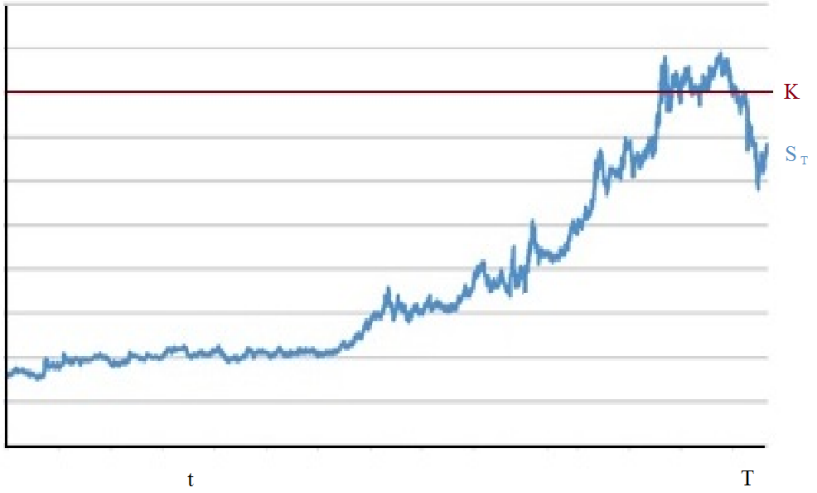
Motivations of Fournier (2015)

- If we suppose that the prices of assets should be "Continuous Martingale of Maximal Variation", we can compute a pricing formula.
- Particularly interesting for options.
  European call: contract which gives the owner the right to buy one unit of S at time T and price K.

  $C^T_{K,t}$:= the value at time t of the call with expiration date T and strike K.

Motivations of Fournier (2015)





$$C^T_{K,T} = S_T - K \qquad\qquad C^T_{K,T} = 0$$

$$\begin{aligned} C^T_{K,T} &= (S_T - K)^+ \\ C^T_{K,t} &= ??? \end{aligned}$$

Fournier (2015): Model with 2 assets $\Gamma_n(\mu)$

- At the beginning of the game, the final value L^1 of a risky asset is drawn with law μ.
- The second risky asset has a final value $f(L^1)$. f is a deterministic and increasing function. Player 1 knows (L^1, L^2). Player 2 only knows the law μ and the function f.

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- At the beginning of the game, the final value L^1 of a risky asset is drawn with law μ.
- The second risky asset has a final value $f(L^1)$. f is a deterministic and increasing function. Player 1 knows (L^1, L^2). Player 2 only knows the law μ and the function f.
- At each round $q \in \{1, \dots, n\}$, players exchange the two assets against money: player 1 selects $(u_q^1, u_q^2) \in \{-1, +1\}^2$ and player 2 selects the prices $(p_q^1, p_q^2) \in \mathbb{R}^2$ of the transactions at stage q.

Behavioral strategies

At period $q \in \{1, \dots, n\}$, players have observed past actions.

- Behavioral strategy of player 1: $\sigma = (\sigma_1, \dots, \sigma_n)$
where $\sigma_k : (h_{k-1}, L^1, L^2) \mapsto \sigma_k(h_{k-1}, L^1, L^2) \in \Delta(\{+1, -1\}^2)$.

- Behavioral strategy of player 2: $\tau = (\tau_1, \dots, \tau_n)$
where $\tau_k : (h_{k-1}) \mapsto \tau_k(h_{k-1}) \in \Delta(\mathbb{R}^2)$.

Main results

Under smoothness hypothesis on μ, H and f,

1. $\Gamma_n(\mu)$ has a Nash equilibrium where:
   - $\rightarrow$ Players play reduced strategies
   - $\rightarrow$ Player 2 plays a pure strategy
   - $\rightarrow$ Player 1 play a mixed symmetric strategy (he takes the same action for both assets at each round)

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Under smoothness hypothesis on μ, H and f,

1. $\Gamma_n(\mu)$ has a Nash equilibrium where:
   - $\rightarrow$ Players play reduced strategies
   - $\rightarrow$ Player 2 plays a pure strategy
   - $\rightarrow$ Player 1 play a mixed symmetric strategy (he takes the same action for both assets at each round)
2. Consider this player 2's equilibrium strategy and suppose that $(u_1^1, u_1^2, \ldots, u_n^1, u_n^2)$ has the same distribution than $(u_1, u_1, \ldots, u_n, u_n)$ when $(u_1, \ldots, u_n) \sim \lambda_n$, then:
   - $\rightarrow$ The price processes are martingales.
   - $\rightarrow$ If players play more and more frequently, the limit process is a Continuous Martingales with Maximal Variations.

Smoothness hypothesis

- μ is a probability measure on a compact interval, absolutely continuous with respect to the Lebesgue measure. Its density function f_μ is strictly positive and C^1.
- H is strictly positive, strictly convex and C^2. H' is Lipschitz and bounded.
- f is strictly increasing and C^2.

Sketch of the proof

1. We use an auxiliary game where players exchange a package of one unit of each risky asset $\simeq G_n(\hat{\mu})$.
2. There exists $(\hat{\Pi}_n, \hat{\Psi}_n)$, an equilibrium in $G_n(\hat{\mu})$.
3. Suppose that player 1 plays the symmetric strategy that duplicate the strategy $\hat{\Pi}_n$. The payoff of player 2 only depends on the sum of prices he selects. At equilibrium, this sum has to be equal to the price associated to $\hat{\Psi}_n$.
4. We have one degree of freedom, we select the prices such that the best answer of player 1 is to play the symmetric strategy that duplicates $\hat{\Pi}_n$.

Pricing with CMMV: The algorithm

Inputs: S_t and $C_{K,0}$ for all K.

Output: $C_{K,t}$.

Step 1 We compute μ the law of S_T.

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$$\begin{aligned}\frac{\partial}{\partial K}(C_{K,0}) =& \frac{\partial}{\partial K}E(C_{K,T})\\ =& E(\frac{\partial}{\partial K}C_{K,T})\\ =& E(\frac{\partial}{\partial K}(S_T-K)^+)\\ =& E(-1_{S_T>K}) = -P(S_T>K)\end{aligned}$$

Step 1 We compute μ the law of S_T.

Step 2 We compute the function $x \to f(x, T)$.

$F_\mu^{-1} \circ F_{B_T}$ is the unique function such that $F_\mu^{-1}(F_{B_T}(B_T)) \sim \mu$:

$$S_T = f(B_T, T) = F_\mu^{-1}(F_{B_T}(B_T))$$

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Step 4 Finally, we can compute $C_{K,t}$:

$$\begin{aligned} C_{K,t} =& E(C_{K,T} \mid \mathcal{F}_t) \\ =& E((S_T - K)^+ \mid \mathcal{F}_t) \\ =& E((f(B_T, T) - K)^+ \mid \mathcal{F}_t) \\ =& E((f(B_t + B_T - B_t, T) - K)^+ \mid \mathcal{F}_t) \\ =& E_Z((f(B_t + Z\sqrt{T-t}, T) - K)^+) \text{ where } Z \sim \mathcal{N}(0,1) \end{aligned}$$

Thank you

Proposition

If B is a Brownian motion on a filtration $(\mathcal{F}_t)$ and g is an increasing function $\mathbb{R} \to \mathbb{R}$, then $E[g(B_1)|\mathcal{F}_t]$ is a CMMV.

$$E[g(B_1)|\mathcal{F}_t] = E[g(B_1)|B_t] = E[g(B_t + (B_1 - B_t))|B_t].$$

$E[g(B_1)|\mathcal{F}_t] = f(B_t, t)$, where $f(x,t) = E_z[g(x + z\sqrt{1-t})]$, with $z \sim \mathcal{N}(0,1)$.

f is the convolution of g with a normal density kernel. This convolution preserves the class of increasing functions and f is thus indeed increasing in x.

A partial solution: a profile of strategies that satisfies **(C1)**, **(C2)** et **(C3)**.

Lemma

*There exists a family of partial solutions: for every probability $\nu \in \Delta(\mathbb{R})$, there exists a couple (Π_ν, Ψ_ν) solution of **(C1)**, **(C2)** and **(C3)**.*

$\star$ Π_ν is the law of $(L, F_\nu^{-1}(F_\mu(L)))$ when $L \sim \mu$.

$\star$ $\Psi_\nu := x \mapsto \int_0^x F_\mu^{-1}(F_\nu(t))dt - E_{\lambda_n}[\int_0^x F_\mu^{-1}(F_\nu(t))dt]$